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FORMULATING THE GENETIC DECOMPOSITION FOR PROBABILITY CONSTRUCTION BASED ON STUDENT DESCONSTRUCTED PROBLEM-SOLVING PROCESS

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Abstract

Probability theory is inherently abstract and structurally complex, often creating substantial cognitive barriers for students when solving problems. This study aims to formulate a genetic decomposition of core probability concepts based on the mental structures constructed by undergraduate learners. Guided by the APOS (Action–Process–Object–Schema) theoretical framework, the research employed a three-phase methodological cycle: (1) epistemological analysis of fundamental probability concepts, (2) design and administration of diagnostic and instructional instruments, and (3) analysis and validation of students' cognitive structures. The study was conducted at two universities in Central Java, Indonesia, involving four mathematics education undergraduates enrolled in an introductory probability course. Data were collected through content analysis, domain-specific assessment instruments, and semi-structured interviews. The APOS framework was used to trace students' mental constructions in relation to classical probability, independence, conditional probability, total probability, and Bayes' theorem. The results reveal that students had not fully internalized the mental objects and schemas required to coordinate these concepts during problem-solving activities. Their reasoning remained largely at the process level, particularly in representing sample spaces, applying counting techniques, and thematizing probability structures. Nonetheless, several elements of their cognitive behavior aligned with the refined genetic decomposition. The findings highlight the necessity of strengthening students' APOS-based mental constructions to support the development of coherent probabilistic schemas and enhance conceptual as well as procedural fluency.

Keywords: APOS, Genetic Decomposition, Mental Mechanism, Mental Structure, Probability

Abstrak

Teori probabilitas memiliki sifat yang abstrak dan struktur yang kompleks sehingga sering menimbulkan hambatan kognitif bagi mahasiswa dalam menyelesaikan masalah. Penelitian ini bertujuan merumuskan dekomposisi genetik konsep-konsep dasar probabilitas berdasarkan struktur mental yang dibangun oleh mahasiswa. Dengan menggunakan kerangka teoretis APOS (Action–Process–Object–Schema), penelitian ini mengikuti tiga tahapan metodologis: (1) analisis epistemologis konsep probabilitas, (2) perancangan serta penerapan instrumen diagnostik dan instruksional, dan (3) analisis serta validasi struktur kognitif mahasiswa. Penelitian dilaksanakan di dua perguruan tinggi di Jawa Tengah, Indonesia, dengan melibatkan empat mahasiswa pendidikan matematika yang mengikuti mata kuliah pengantar probabilitas. Data diperoleh melalui analisis konten, tes berbasis konsep probabilitas, dan wawancara semi-terstruktur. Analisis dengan teori APOS digunakan untuk menelusuri konstruksi mental mahasiswa terkait probabilitas klasik, kejadian bebas, probabilitas bersyarat, probabilitas total, dan teorema Bayes. Hasil penelitian menunjukkan bahwa mahasiswa belum sepenuhnya menginternalisasi objek dan skema kognitif yang diperlukan untuk mengoordinasikan konsep-konsep tersebut dalam proses pemecahan masalah. Sebagian besar konstruksi mereka masih berada pada level proses, terutama dalam merepresentasikan ruang sampel, menerapkan teknik enumerasi, dan mentematisasi struktur

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probabilitas. Meskipun demikian, beberapa aspek perilaku matematis mahasiswa menunjukkan kesesuaian dengan dekomposisi genetik yang telah direvisi. Temuan ini menegaskan pentingnya penguatan konstruksi mental berbasis APOS untuk mendukung pembentukan skema probabilitas yang koheren serta meningkatkan kefasihan konseptual dan prosedural.

***Kata kunci**: APOS, Genetic decomposition, Mental mechanism, Mental structure, Probability*

INTRODUCTION

Probability theory constitutes a fundamental component of the undergraduate mathematics curriculum in many countries, including Ghana, Spain, China, Germany, and the United States. Education systems have incorporated probability into primary and secondary curricula due to its role in developing basic statistical literacy on a global scale (Morsanyi et al., 2013). Mathematics teacher education programs expect pre-service teachers to acquire strong probabilistic reasoning and problem-solving competencies. Beyond its relevance in daily decision-making, probability also serves as a foundational framework for multiple scientific disciplines and enhances students' mathematical modeling skills ((Arum et al., 2018); (Rahman & Ahmar, 2016)).

Probability instruction frequently engages students in problem contexts drawn from real-world phenomena, requiring the application of probabilistic models. Despite its curricular importance, many undergraduates encounter substantial cognitive challenges when learning probability. Studies have documented widespread misconceptions in probabilistic concepts Masel et al. (2015), Triliana and Asih (2019), which contribute to flawed probabilistic reasoning. These conceptual difficulties include misinterpretation of sample spaces, incorrect identification of independent and dependent events, and limited understanding of distinctions between discrete and continuous distributions (Sezgin-Memnun et al., 2019).

Procedural errors often stem from students' failure to correctly implement combinatorial techniques, probability rules, or algebraic manipulations—despite partial conceptual understanding (Arum et al., 2018). Additionally, students encounter interpretational challenges when they misrepresent probabilistic events or fail to contextualize results ((Konold, 2017); (Pisarenko, 2018)). These cognitive barriers result in performance levels below the expected mastery threshold in probability assessments (Estrada et al., 2018).

Given recent pedagogical developments in teaching probability, researchers must examine how university students cognitively construct probabilistic knowledge. Tracing students' mental constructions is essential for supporting schema development and promoting both procedural and conceptual fluency. Researchers must identify the epistemological obstacles and algorithmic errors students encounter during probabilistic problem-solving (Bintoro et al., 2021). These findings can inform the design of diagnostic and instructional instruments that support meaningful learning.

To analyze cognitive structures in probability, researchers frequently employ the APOS theory (Action–Process–Object–Schema). APOS models mathematical knowledge as a dynamic system of mental construc-tions, where learners build and reorga-nize actions, processes,

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and objects into coherent schemas through reflection and social mediation (Dubinsky & Mcdonald, 2001). APOS theory defines six interior mechanisms—interiorization, coordina-tion, reversal, encapsulation, deencap-sulation, and thematiza-tion—which describe the develop-ment of students' cognitive structures (Arnon et al., 2014). These mechanisms offer a framework for understanding students' schema development in mathematical contexts (Olesova & Borisova, 2016).

A central analytical tool within APOS theory is genetic decomposition, which predicts how learners might construct a mathematical concept through a sequence of coordinated mental transformations. Genetic decom-position constitutes a theoretically informed cognitive blueprint that identifies essential mental actions, their transformation into internal processes, and the encapsulation of these processes into mathematical objects ((Tarr & Maharaj, 2021); (Widada, 2017)). Researchers typically derive genetic decompositions through epistemological analysis, literature synthesis, teaching experience, or a combination of these methods (Salgado & Trigueros, 2015).

Unlike step-by-step instructional sequences or lists of misconceptions, a genetic decomposition delineates the mental constructions necessary for learners to internalize and transform mathematical ideas. It describes how specific actions on mathematical objects evolve into processes through interiorization and how these processes become encapsulated into objects. These constructions then interrelate and organize into higher-level schemas. A single concept may involve multiple interwoven mental structures, each contributing to the overall schema governing that concept.

Previous research has successfully applied APOS theory and genetic decomposition to model concept formation in calculus, abstract algebra, inferential statistics, logic, linear algebra, real analysis, and discrete mathematics (Arnon et al., 2014).

Wijayanti (2017) proposed preli-minary and refined genetic decompo-sitions in group theory. Stalvey et al. (2019) developed a decomposition for hypothesis testing. Zwanch (2019) out-lined a decomposition for probabilistic independence. However, prior studies have not systematically addressed the genetic decomposition of core probability concepts.

This study constructs a genetic decomposition for undergraduate-level probability topics, including classical probability, probabilistic independence, conditional probability, the law of total probability, and Bayes' theorem. The researchers employed the APOS framework to guide the construction and analysis of this decomposition. The resulting cognitive model offers instructional insights for designing learning environments that strengthen students' probabilistic reasoning and support the development of robust conceptual schemas.

METHODS

This study adopted the APOS theoretical framework to guide a three-phase research cycle: (1) epistemo-logical analysis, (2) design and imple-mentation of diagnostic and instruc-tional instruments, and (3) analysis and validation of students' cognitive structures. This cyclical methodology enabled the researchers to investigate how undergraduates construct core probabilistic schemas, particularly by identifying their underlying mental structures and genetic mechanisms.

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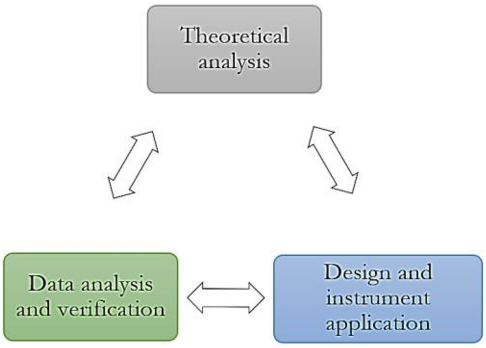

Figure 1. The investigative methodology cycle of APOS theory (Asiala et al., 1998)

In the first phase, the researchers conducted an epistemological decomposition of foundational probability concepts to hypothesize a genetic decomposition. They constructed this decomposition by synthesizing insights from textbook analysis, prior teaching experiences, and common algorithmic and interpretational challenges in student learning. In the second phase, they operationalized this initial decomposition into classroom instruction by embedding it within a sequence of task-based learning activities.

In the third phase, they evaluated the adequacy of the preliminary genetic decomposition as a model for the schema development of key probabilistic constructs, including classical probability, probabilistic independence, conditional probability, total probability, and Bayes' theorem. They analyzed the mental actions, processes, and objects constructed by students while solving contextual probability problems, using these findings to refine the decomposition. After implementing the refined version in instruction, they administered problem-solving assessments and conducted semi-structured interviews to validate the students' resulting cognitive structures and schema transformations.

The study involved 46 mathematics education students from Universitas Muria Kudus (Indonesia) and Universitas Muhammadiyah Purwokerto (Indonesia) enrolled in Introduction to Probability. All participants completed a conceptual-procedural test on probability, and the researchers analyzed their written responses using APOS theory. The researchers selected four students for cognitive interviews based on the completeness of their problem-solving traces, clarity in their probabilistic reasoning, and the uniqueness of their cognitive schema activation.

The researchers triangulated data using (1) documented test solutions, (2) constructed-response items, and (3) guided cognitive interviews. The written test consisted of five open-ended items targeting procedural fluency and conceptual understanding in probability. Interview protocols elicited students' mental constructions and cognitive transitions—including evidence of interiorization, encapsulation, reversal, and thematization, as described in APOS theory. Experts reviewed both instruments to ensure content validity. Reliability testing using Cronbach's alpha produced a coefficient of 0.611, indicating that the assessment instrument achieved high internal consistency.

The researchers analyzed student data through APOS-based cognitive structure validation. They aligned the results of their triangulation procedures with the hypothesized genetic decomposition. The first triangulation compared written responses to interview data, identifying key APOS elements, such as action-process-object conversions and evidence of schema integration. The second triangulation compared these findings to the theore-

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tical decomposition derived from epistemological analysis, thereby connecting the conceptual structure of probability with an instructional framework grounded in cognitive theory.

RESULTS AND DISCUSSION

This study implemented an APOS-based research methodology, comprising three iterative phases: (1) theoretical analysis, (2) design and implementation of diagnostic and instructional instruments, and (3) data analysis and cognitive structure validation.

Theoretical Analysis

In the first phase, the researchers analyzed the conceptual structure of probability theory to construct a hypothetical genetic decomposition. This decomposition models the sequence of mental constructions students must develop to internalize probabilistic concepts. Following Ortiz and Parraguez (2014), the researchers identified the key conceptual components of probability as random experiments, events, discrete sample spaces, counting techniques (e.g., permutations and combinations), probability axioms, conditional probability, probabilistic independence, the law of total probability, and Bayes' theorem. Figure 2 provides the initial version of the genetic decomposition.

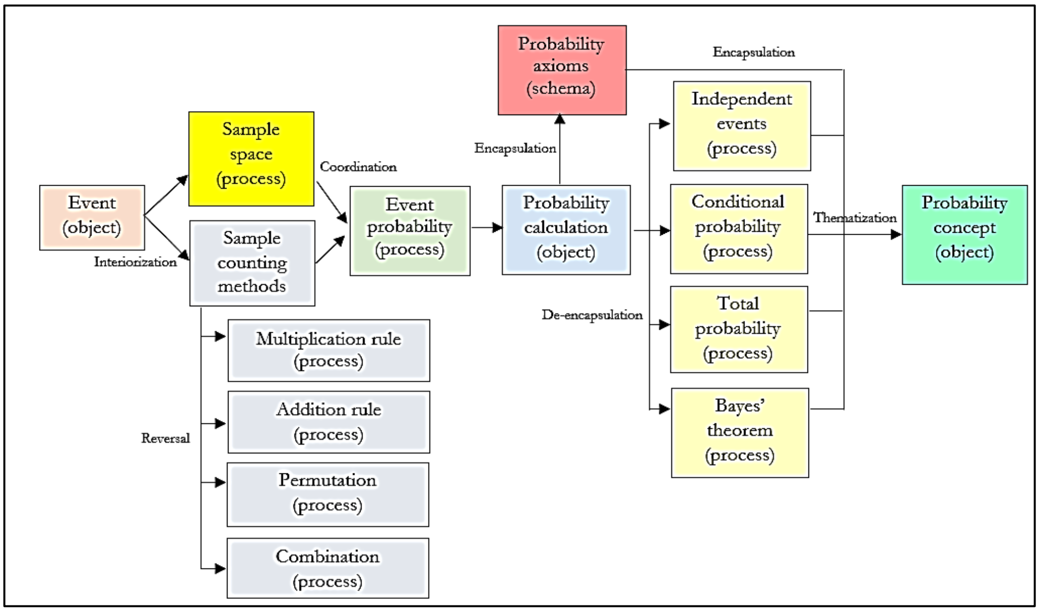

Figure 2. The hypothetical genetic decomposition of probability concept

Design and Implementation

In the second phase, the researchers constructed instructional and diagnostic instruments aligned with the preliminary genetic decomposition of probability. This decomposition reflects the developmental trajectory of students' mental structures and cognitive mechanisms, as proposed by APOS theory. The researchers grounded the decomposition in:

1) the conceptual composition of the probability domain
2) the hypothesized sequence of Action–Process–Object–Schema (APOS) constructions, and
3) the obtained pedagogical experience in teaching probability.

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The domain composition includes five core concepts: classical probability, independence, conditional probability, total probability, and Bayes' theorem (He, 2025). Based on classroom experience, the researchers observed that students typically begin by determining the sample space and computing sample points using the addition rule, multiplication rule, permutations, and combinations. Students then proceed through the following cognitive sequence: 1) classical probability: understanding equally likely outcomes, applying the classical formula, and interpreting the probability interval [0,1]; 2) independent events: computing and reasoning about joint probabilities and testing for probabilistic independence; 3) conditional probability: applying definitions and formulas and interpreting dependence relationships; 4) total probability: integrating mutually exclusive and exhaustive events; and 4) Bayes' theorem: using posterior probabilities derived from conditional and total probability.

From this analysis, the researchers concluded that the overarching probability schema consists of five interrelated sub-schemas: (1) the classical probability schema, (2) the independence schema, (3) the conditional probability schema, (4) the total probability schema, and (5) the Bayesian schema.

They synthesized the epistemological analysis, domain composition, and instructional insights to formulate a preliminary genetic decomposition of probability, detailed in the Table 1.

Table 1. The preliminary genetic decomposition about probability material

Concept	Preliminary Genetic Decomposition
Classical Probability	1: In an action, when confronted with an event A, the individual identifies the sample space SSS and calculates its cardinality n(S) 2: Through interiorization, the individual performs a process that involves: a) identifying the sample points constituting event A, b) Calculating n(A), the number of favorable outcomes. 3: The individual then encapsulates the second process into an object, the classical probability of event. 4: The individual thematizes the third steps into a probability schema based on the axiomatic condition $0 \leq P(A) \leq 1$. 5: The individual then applies the complete concept of classical probability to specific event scenarios.
Independent events	1: In an action, when given two events A and B, the individual defines the sample space and computes n(S). 2: Through interiorization, the individual engages in a process that involves: a. identifying the sample points of A and B, b. determining n(A) and n(B), c. computing P(A) and P(B). 3: The individual reconstructs the process to determine the intersection A∩B and evaluates P(A∩B). 4: The individual then encapsulates the second and third processes of independence as an object). 5: The individual thematizes the 4P into a schema of probabilistic independence. 6: The individual applies the complete independence concept to analyze specific contexts.

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Concept	Preliminary Genetic Decomposition
Conditional Probability	1: In an action, given events A and B, the individual defines the sample space and calculates n(S). 2: Through interiorization, the individual carries out the first process involving: a) identifying the sample points of A, B, and A∩B, b) computing n(A), n(B), and n(A∩B), c) calculating P(A), P(B), and P(A∩B). 3: The individual reconstructs the second process into the concept of conditional probability, by providing the value of conditional probability. 4: The individual encapsulates the second and third processes to manipulate the formula of conditional probability from $P(B\|A) = \frac{P(A\cap B)}{P(A)}$ into an object. 5: The individual thematizes the fourth step into a schema of conditional probability. 6: The individual applies the complete concept of conditional probability to more specific, dependent event cases.
Total Probability	1: In an action, when given some events, the individual defines the sample space and determines n(S). 2: Through interiorization of step 1, the individual performs a process that includes: a) identifying intersections o the event samples and the intersections. b) computing the relevant sample sizes and probabilities, c) evaluating conditional probabilities. 3: The individual reconstructs the second step into total probability by providing the total probability value. 4: The individual encapsulates the second and third steps to manipulate the formula of total sampling into object. 5: The individual thematizes the 4P step into a schema of total probability. 6: The individual applies the complete total probability concept in situations involving specific events.
Bayes' Theorem	1: In an action, given several events, the individual constructs representations of the events. 2: Through interiorization of step 1, the individual initiates a process involving: a) identifying the sample points and their intersections, b) computing all relevant sample event points from the first step, c) computing the values of event probability and conditional probability. 3: The individual reconstructs the second of total sampling by providing the total probability value. 4: The individual encapsulates the second and third steps to manipulate the Bayes' theorem into object. 5: The individual thematizes the fourth step into a schema of Bayes' theorem. 6: The individual applies the complete Bayes' theorem to specific events.

Analysis and Data Verification

Researchers analyzed and verified the preliminary genetic decomposition to assess its validity as a construction model for the concept of probability. Seven experts conducted a validation

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test, yielding an average score of 3.46, which falls into the "strong" category. The validation results indicated that (1) the mental structures aligned with the conceptual content, (2) they corresponded with the associated mental mechanisms, (3) the genetic decomposition matched the mental structures that students need to develop probabilistic understanding, and (4) the language used in the decomposition was clear, grammatically correct, and avoided ambiguity.

The second phase of analysis and verification involved administering the first-stage problem-solving task. After students completed the probability tasks, researchers conducted in-depth interviews to investigate the mental structures and mental mechanisms that students had constructed. These interviews served as a triangulation tool to validate the preliminary decomposition. The interview transcripts provided a detailed representation of students' cognitive processes, which researchers used to refine the initial genetic decomposition.

The students' problem-solving results, along with the mental structures and mechanisms they constructed, revealed that most participants remained at the process level of mental structure for each probabilistic topic (Rahayu & Agoestanto, 2023). Researchers then synthesized these results and compared them with the preliminary genetic decomposition to assess alignment.

Table 2 summarizes the degree of alignment between students' responses in the first-stage problem-solving test and the initial genetic decomposition. The table indicates that students' mental construction of probabilistic concepts did not yet match the expected mental structures defined in the genetic decomposition model.

Table 2. The first step result correspondence with preliminary genetic decomposition

Mental Structure Indicator	**The Deconstructed Mental Structure Of The Students**				**Genetic Decomposition Correspondence**
	Subject1	**Subject2**	**Subject3**	**Subject4**	
Construct the event representation (Action)	Action	Action	Action	Action	Mostly correspond
Represent the sample space (Process)	Process	Process	Mostly process	Mostly process	Mostly not correspond
Select a sample point enumeration method (Object)	Object	Object	Object	Object	Mostly correspond
Apply the classical probability formula (Process)	Process	Process	Process	Process	Mostly correspond
Execute probability calculation procedures (Object)	Process	Process	Mostly process	Mostly process	Mostly not correspond
De-encapsulate the concept of probabilistic independence during the probability computation (Process)	Process	Process	Process	Process	Mostly correspond
De-encapsulate the concept of conditional probability during the probability computation (Process)	Process	Process	Mostly process	Process	Mostly not correspond
De-encapsulate the concept of total probability during the probability computation (Process)	Process	Process	Process	Mostly process	Mostly not correspond

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Mental Structure Indicator	The Deconstructed Mental Structure Of The Students				Genetic Decomposition Correspondence
	Subject1	Subject2	Subject3	Subject4	
De-encapsulate Bayes' theorem during the probability computation (Process)	Process	Process	Process	Mostly process	Mostly not correspond
Evaluate the probability axioms (Scheme)	Scheme	Scheme	Scheme	Mostly scheme	Mostly not correspond
Construct the mental structure of the probability concept.					Mostly not coherent

The results from Phase 1, which examined students' mental constructions while solving probability tasks, identified a misalignment between their constructed schema and the mental structures hypothesized in the preliminary genetic decomposition. This incongruence indicated the necessity of refining the initial genetic decomposition model. Most students exhibited constructions limited to the process level and had not yet encapsulated their understanding into coherent objects or an integrated schema for the probability concept. Table 3 summarizes the cognitive discrepancies and presents the refinements applied to the genetic decomposition based on the epistemological analysis of students' performance.

Table 3. The student performance differences and genetic decomposition revisions

No.	Student Performance Differences	Genetic Decomposition Revisions
1	Based on the test and interview results, students first performed an action by reading the problem. They found pertinent information and broke down the issue into its most basic elements by interiorizing.	Separating the action into two stages: (a) reading and understanding the problem and (b) applying the mental mechanism of interiorization.
2	Before they encapsulated the process into an object, students engaged in reversal by recalling prior knowledge to support problem-solving.	Exploring the full range of possible reversal mechanisms involved.
3	Students performed de-encapsulation by restructuring their understanding of Bayes' theorem, which integrates the concept of conditional probability.	Adding a mental mechanism of de-encapsulation related to computing event probabilities and conditional probabilities.

The next step involved refining the preliminary genetic decomposition, functioning as a revised cognitive model for the construction of the probability concept. This refinement addressed the misalignment between the preliminary genetic decomposition and students' actual cognitive structures, as identified through analysis of their problem-solving processes, mental mechanisms (e.g., interiorization, encapsulation, de-encapsulation), and APOS-level constructions. Table 4 presents both the initial genetic decomposition and its refined version, which better reflects students' schema development in probability.

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Table 4. The genetic decomposition refinement on probability materials

Concepts	The Refinement of Preliminary Genetic Decomposition
Classical Probability	1: Students perform an action by reading and understanding the situation involving events A and B. 2: They interiorize the action of step 1 to construct the sample space and determine its cardinality n(S). 3: Students interiorize the second step further into a process consisting of: a. Identifying the sample points for a given event. b. Calculating the cardinality of the event space n(A) . 4: They apply reversal using methods for calculating sample points, including the multiplication rule, addition rule, permutations, and combinations. 5: They coordinate these third and fourth processes to compute the classical probability. 6: Students encapsulate the fifth process into an Object, assuming action to apply. 7: They thematize this object of sixth step into a probability schema, satisfying the condition of conditional probability value $0 \leq P(A) \leq 1$. 8: They apply the complete concept of probability to specific events.
Independent Events	1: Students read and understand the situation involving two events, A and B (Action). 2: They interiorize the action of step 1 to construct the sample space and determine the cardinality of the sample space, n(S). 3: They coordinate the actions of first and second processes to form a process consisting of a. Identifying sample points for events A and B. b. Calculating the number of sample points for events n(A) and n(B). 4: They apply reversal using methods: multiplication rule, addition rule, permutations, and combinations. 5: They coordinate the third and fourth processes to determine the values of event probabilities P(A) and P(B). 6: They reconstruct the process of the fifth step of $A \cap B$ event by providing the probability value of $A \cap B$. 7: They perform de-encapsulation the rule of independent events. 8: They encapsulate the process of fifth and sixth steps to manipulate the independent event probability into object. 9: They thematize the object of step 7 into a schema of probabilistic independence. 10: They apply the concept of independent event probability completely into specific events.
Conditional Probability	1: Students read and understand the situation involving events A and B (action). 2: They interiorize the action of step 1 to construct the **sample space** and determine the cardinality of sample space n(S). 3: They interiorize the first and second steps further into a process, consisting of a. Identifying sample points for A, B, and $A \cap B$. b. Calculating the cardinality of event sample points n(A) and n(B) 4: They interiorize the first and second steps further into a process, consisting of a. Identifying sample points for A, B, and $A \cap B$. b. Calculating the cardinality of event sample points n(A) and n(B) 5: They apply reversal with counting methods of sample points, starting from multiplication, addition, permutation and combination.

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Concepts	The Refinement of Preliminary Genetic Decomposition
	6: They coordinate the second and thirds steps to count the event probability values $P(A), P(B), dan\ P(A \cap B)$.
	7: They reconstruct the process of the fifth step for the event's conditional probability by providing the conditional probability value.
	8: They encapsulate the process of fifth and sixth steps to manipulate conditional probability formula $P(B\|A) = \frac{P(A \cap B)}{P(A)}$ into an object.
	9: They thematize the object of fourth step into a conditional probability schema.
	10: They completely apply conditional probability to specific problems.
Total Probability	1: Students read and understand the situation involving events A and B (action).
	2: They interiorize the first step to construct the sample space and determine cardinality of the sample space n(S).
	3: They interiorize the action of step 1 further into a process, consisting: a. Identifying sample points and their intersections. b. Calculating sample point cardinalities of step (a).
	4: They coordinate the second and thirds steps to determine probabilities.
	5: They reconstruct the processes of third and fourth steps into conditional probability by showing the total probability value.
	6: They encapsulate the processes of fourth and fifth steps to manipulate the total probability into an object.
	7: They thematize the object of the fourth step into a total probability schema.
	8: They apply total probability specific events.
Bayes' Theorem	1: Students read and understand the situation involving multiple events (action).
	2: They interiorize the action of step 1 to define event representations.
	3: They coordinate the step 1 into a process, consisting of: a. Identifying sample points and intersections. b. Calculating cardinalities of the sample points based on step (a).
	4: They perform de-encapsulation the calculation of event probability values and conditional probability
	5: They construct the process of fourth step with probability axiom to determine the total probability.
	6: They coordinate between event probability, conditional probability, and total probability.
	7: They encapsulate the process of the fifth step to manipulate the Bayes' Theorem into object.
	8: They thematize the object of sixth step into a Bayes' theorem schema.
	9: They apply Bayes' Theorem to specific probability events.

After refining the genetic decomposition, the researcher implemented it as the instructional model for probability concepts. Upon completion of ins-truction, students undertook the second step, problem-solving task. The researcher then conducted in-depth interviews to verify the students' constructed mental structures related to classical probability, probabilistic independence, conditional probability, the law of total probability, and Bayes' theorem.

The analysis revealed that students had not fully constructed the mental structures anticipated in the refined genetic decomposition during their problem-solving processes. The schema constructed by the four subjects was mostly incoherent with the expected schema for classical probability and independent events.

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However, their constructions of conditional probability, total probability, and Bayes' theorem were mostly coherent with the intended schema.

Despite these discrepancies, student performance remained consistent with the mental structures embedded in the refined genetic decomposition. This suggests that effective learning of probability concepts requires students to construct the underlying Action–Process–Object–Schema (APOS) structures reflected in the refined genetic decomposition implemented in this study. Table 5 presents a synthesis of the coherence between students' responses in Phase 2 and the refined genetic decomposition.

Table 5. The corresponding results of step 2 and the genetic decomposition refinement

Mental Structure Indicators	The Deconstructed Mental Structures by Students				Genetic Decomposition Correspondence
	Subject1	Subject2	Subject3	Subject4	
Classical Probability					
Construct the event representation (Action)	Action	Action	Action	Action	Mostly correspond
Represent the sample space (Process)	Process	Mostly process	Process	Mostly process	Mostly not correspond
Select a sample point enumeration method (Object)	Object	Mostly process	Object	Mostly Object	Mostly not correspond
Apply the classical probability formula(Process)	Process	Mostly Action	Process	Process	Mostly not correspond
Execute probability calculation procedures (Object)	Process	Mostly Action	Process	Process	Mostly not correspond
Evaluate the probability axioms (Scheme)	Scheme	Mostly process	Scheme	Mostly Scheme	Mostly not correspond
Construct the mental structure of the probability concept.					Mostly not coherent
Independent Event Probability					
Construct the event representation (Action)	Action	Action	Action	Action	Mostly correspond
Represent the sample space (Process)	Process	Process	Process	Mostly Action	Mostly not correspond
Select a sample point enumeration method (Object)	Mostly process	Mostly Object	Object	Mostly Action	Mostly not correspond
Determining the computing method of two-event intersection (Object)	Mostly process	Object	Object	Mostly Action	Mostly not correspond
Apply the classical probability formula(Process)	Process	Process	Process	Mostly process	Mostly correspond
Execute probability calculation procedures (Object)	Object	Object	Object	Mostly Object	Mostly correspond
De-encapsulate the concept of probabilistic independence during the probability computation(Process)	Mostly Action	Mostly process	Process	Mostly process	Mostly not correspond
Checking the independent events (Scheme)	Mostly Action	Mostly Object	Scheme	Mostly process	Mostly not correspond
Construct the mental structure of the probability concept of independent events					Mostly not coherent

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Mental Structure Indicators	**The Deconstructed Mental Structures by Students**				**Genetic Decomposition Correspondence**
	Subject1	**Subject2**	**Subject3**	**Subject4**	
Conditional Probability					
Construct the event representation (Action)	Action	Action	Action	Action	Mostly correspond
Represent the sample space(Process)	Process	Process	Process	Process	Mostly correspond
Select a sample point enumeration method (Object)	Object	Object	Object	Object	Mostly correspond
Determining the computing method of two-event intersection (Object)	Mostly process	Process	Process	Process	Mostly correspond
Apply the classical probability formula (Process)	Process	Process	Process	Process	Mostly correspond
Execute probability calculation procedures (Object)	Mostly process	Object	Object	Object	Mostly not correspond
De-encapsulate the concept of conditional probability during the probability computation (Process)	Mostly process	Process	Process	Process	Mostly correspond
Checking the concept of conditional probability (Scheme)	Mostly process	Scheme	Scheme	Mostly Scheme	Mostly not correspond
Construct the mental structure of the probability concept of conditional probability					Mostly coherent
Total Probability					
Construct the event representation (Action)	Action	Action	Action	Action	Mostly correspond
Represent the sample space (Process)	Process	Process	Process	Process	Mostly correspond
Select a sample point enumeration method (Object)	Object	Object	Object	Object	Mostly correspond
Apply the classical probability formula (Process)	Process	Process	Process	Process	Mostly correspond
Execute probability calculation procedures (Process)	Process	Process	Process	Mostly process	Mostly correspond
De-encapsulate the concept of conditional probability during the probability computation (Process)	Process	Process	Process	Process	Mostly correspond
De-encapsulate the concept of total probability during the probability computation (Process)	Process	Process	Process	Process	Mostly correspond
Thematizing the total probability (Scheme)	Scheme	Scheme	Scheme	Mostly Scheme	Mostly correspond
Construct the mental structure of the probability concept. total					Mostly coherent
Bayes' Theorem					
Construct the event representation (Action)	Action	Action	Action	Action	Mostly correspond
Represent the sample space (Process)	Process	Process	Process	Process	Mostly correspond

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Mental Structure Indicators	**The Deconstructed Mental Structures by Students**				**Genetic Decomposition Correspondence**
	Subject1	**Subject2**	**Subject3**	**Subject4**	
Select a sample point enumeration method (Object)	Object	Object	Object	Object	Mostly correspond
Determining the computing method of two-event intersection (Object)	Object	Object	Object	Object	Mostly correspond
Apply the classical probability formula (Process)	Process	Process	Process	Process	Mostly correspond
Execute probability calculation procedures (Object)	Object	Object	Object	Mostly Object	Mostly correspond
De-encapsulate the concept of conditional probability during the probability computation (Process)	Process	Process	Process	Process	Mostly correspond
De-encapsulate the concept of total probability during the probability computation (Process)	Process	Process	Process	Mostly process	Mostly correspond
De-encapsulate Bayes' theorem during the probability computation (Process)	Process	Process	Process	Process	Mostly correspond
Evaluate the probability axioms (Scheme)	Scheme	Scheme	Scheme	Scheme	Mostly correspond
Constructing the mental structure of Bayes' theorem					Mostly coherent

This study constructed a genetic decomposition of probability, encompassing classical probability, probabilistic independence, conditional probability, total probability, and Bayes' theorem. The results expanded the prior decomposition, according to Zwanch (2019), particularly by formalizing the genetic decomposition for independent events. (Norton et al., 2025) explain the epistemological framework of probability involves describing, quantifying, modeling, and explaining the process of random sampling. Following this characterization, students must construct appropriate cognitive structures to reason about probabilistic situations. The theoretical analysis initiated the formulation of a preliminary genetic decomposition hypothesis.

The researchers examined this hypothesis by administering a problem-solving assessment and conducting semi-structured interviews to identify and validate the students' mental constructions. They analyzed students' solutions and interview responses to determine whether their constructed cognitive processes aligned with the genetic decomposition. These interviews served as a diagnostic instrument to evaluate the coherence between students' mental structures and the targeted decomposition (Fuentealba et al., 2017; Salgado & Trigueros, 2015).

Step 1 findings showed that students' constructions did not fully align with the preliminary genetic decomposition, particularly during the process-level reasoning. These results necessitated a refinement of the decomposition. This finding corroborates

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Arnon et al. (2014), who emphasized revising genetic decompositions when students' problem-solving strategies deviate from the theoretical model. The researchers restructured the decomposition by integrating fundamental action–process mental mechanisms demonstrated by the students. In this way, the revised genetic decomposition provides a basis for designing instructional sequences appropriate for university-level probability learning (García-Martínez & Parraguez, 2017).

The researchers then used the refined genetic decomposition to structure instructions based on probability. Post-instruction analysis indicated that students had not fully encapsulated the mental structures specified in the refinement. Specifically, students struggled to thematize the schema for classical probability and probabilistic independence in a coherent manner, although they successfully constructed the schemas for conditional probability, total probability, and Bayes' theorem. One key difficulty was students' failure to construct the object for sample space enumeration and event identification. Without this object, students could not identify all possible outcomes, leading to sequential errors in subsequent processes. Despite these challenges, students' performance remained structurally consistent with the refined genetic decomposition. To develop conceptual and procedural fluency in probability, students must construct the cognitive structures embedded in the revised genetic decomposition formulated in this study.

Genetic decomposition models the hierarchical construction of mental structures necessary to learn mathematical concepts. It is not a list of procedural steps or isolated misconceptions (Arnon et al., 2014), but a model of how new mathematical ideas emerge as cognitive transformations of prior concepts. Genetic decomposition provides a coherent mental architecture to explain how mathematical concepts evolve in the learner's mind (Tarr & Maharaj, 2021). In this study, students articulated logically consistent justifications while coordinating action–process–object–schema (APOS) structures in their reasoning (Baye et al., 2021). The APOS framework supports both theoretical modeling and the development of instructional materials, enabling a learning cycle in which mental constructions and corresponding tasks are iteratively refined (Martínez-Planell & Trigueros, 2019).

CONCLUSION AND SUGGESTION

The genetic decomposition of probability concepts comprises classical probability, independent events, conditional probability, the law of total probability, and Bayes' theorem. Students did not fully construct the expected mental structures within the refined genetic decomposition during the problem-solving process. The Action–Process–Object–Schema (APOS) analysis revealed that the mental constructions of the four participants were mostly incoherent with the target schemas for classical probability and independence of events. In contrast, their constructions for conditional probability, total probability, and Bayes' theorem were mostly coherent with the corresponding schemas. Despite the observed gaps in schema development, students' performance did not significantly deviate from the structure of the refined genetic decomposition. To develop procedural and conceptual fluency in probability, students must construct the mental structures outlined in the refined

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decomposition implemented in this study. This refined genetic decomposition may serve as a preliminary decomposition for future research grounded in the APOS theoretical framework.

REFERENCES

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS Theory*. Springer. https://doi.org/10.1007/978-1-4614-7966-6

Arum, D. P., Kusmayadi, T. A., & Pramudya, I. (2018). Students' difficulties in probabilistic problem-solving. *Journal of Physics: Conference Series*, *983*. https://doi.org/10.1088/1742-6596/983/1/012098

Asiala, M., Brown, A., Kleiman, J., & Mathews, D. (1998). The development of students' understanding of permutations and symmetries. *International Journal of Computers ForMathematical Learning*, *3*(1), 13–43.

Baye, M. G., Ayele, M. A., & Wondimuneh, T. E. (2021). Implementing GeoGebra integrated with multi-teaching approaches guided by the APOS theory to enhance students' conceptual understanding of limit in Ethiopian Universities. *Heliyon*, *7*(e07012), 1–13. https://doi.org/10.1016/j.heliyon.2021.e07012

Bintoro, H. S., Sukestiyarno, Y. L., Mulyono, & Walid. (2021). The Spatial Thinking Process of the Field-Independent Students based on Action-Process-Object-Schema Theory. *European Journal of Educational Research*, *10*(4), 1807–1823. https://doi.org/10.12973/eu-jer.10.4.1807

Dubinsky, E. D., & Mcdonald, M. A. (2001). APOS : A Constructivist Theory of Learning in Undergraduate Mathematics Education Research. In *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 275–282).

Estrada, A., Batanero, C., & Diaz, C. (2018). Exploring teachers' attitudes towards probability and its teaching. In *Teaching and learning stochastics* (In C. Bata, pp. 313–332). Springer. https://doi.org/10.1007/978-3-319-72871-1_18

Fuentealba, C., Sánchez-Matamoros, G., Badillo, E., & Trigueros, M. (2017). Thematization Of Derivative Schema in University Students: Nuances in Constructing Relations Between a Function's Successive Derivatives. *International Journal of Mathematical Education*, *48*(3), 374–392. https://doi.org/doi.org/10.1080/0020739X.2016.1248508

García-Martínez, I., & Parraguez, M. (2017). The basis step in the construction of the principle of mathematical induction based on APOS theory. *Journal of Mathematical Behavior*, *46*, 128–143. https://doi.org/10.1016/j.jmathb.2017.04.001

He, S. (2025). How do preservice mathematics teachers analyze and respond to student errors in solving probability problems using tree diagrams? *Teaching and Teacher Education*, *168*, 105250.

ISSN 2089-8703 (Print)
ISSN 2442-5419 (Online)

DOI: https://doi.org/10.24127/ajpm.v14i4.14367

https://doi.org/https://doi.org/10.1016/j.tate.2025.105250

Konold, C. (2017). Issues in assessing conceptual understanding in probability and statistics. *Journal of Statistics Education*, *3*(1), 1–9. https://doi.org/10.1080/10691898.1995.11910479

Martínez-Planell, R., & Trigueros, M. (2019). Using cycles of research in APOS: The case of functions of two variables. *Journal of Mathematical Behavior*, *55*(100687), 1–22. https://doi.org/10.1016/j.jmathb.2019.01.003

Masel, J., Humphrey, P. T., Blackburn, B., & Levine, J. A. (2015). Evidence-based medicine as a tool for undergraduate probability and statistics education. *CBE Life Sciences Education*, *14*(4), 1–10. https://doi.org/10.1187/cbe.15-04-0079

Morsanyi, K., Handley, S. J., & Serpell, S. (2013). Making heads or tails of probability: An experiment with random generators. *Journal OfEducational Psychology*, *3*, 379–395. https://doi.org/10.1111/j.2044-8279.2012.02067.x

Norton, A., Antonides, J., Arnold, R., & Kokushkin, V. (2025). Logical implications as mathematical objects: Characterizing epistemological obstacles experienced in introductory proofs courses. *The Journal of Mathematical Behavior*, *79*, 101253. https://doi.org/https://doi.org/10.1016/j.jmathb.2025.101253

Olesova, A. P., & Borisova, U. S. (2016). Formation of Professional-communicative Competence of the Future Teachers in The Conditions of the Yakut-Russian Bilingualism. *International Electronic Journal of Mathematics Education*, *11*(10), 3435–3445.

Ortiz, C. V., & Parraguez, M. (2014). Construcciones mentales para el aprendizaje del concepto de probabilidad : un estudio de caso. *Educación Matemática*, *26*(3), 37–74.

Pisarenko, V. F. (2018). The notion of probability and difficulties of interpretation. *Herald of the Russian Academy of Sciences*, *88*(4), 289–293. https://doi.org/10.1134/S1019331618040056

Rahayu, R., & Agoestanto, A. (2023). Problem-Solving Process of Students with a Reflective Cognitive Style Based on the Action-Process-Object-Schema Theory. In *European Journal of Educational* …. pdf.eu-jer.com. https://pdf.eu-jer.com/EU-JER_12_1_41.pdf

Rahman, A., & Ahmar, A. S. (2016). Exploration of Mathematics Problem Solving Process Based on The Thinking Level of Students in Junior High School. *International Journal of Environmental & Science Education*, *11*(14), 7278–7285.

Salgado, H., & Trigueros, M. (2015). Teaching Eigenvalues and Eigenvectors Using Models and APOS Theory. *Journal of Mathematical Behavior*, *39*, 100–120. https://doi.org/10.1016/j.jmathb.2015.06.005

Sezgin-Memnun, D., Ozbilen, O., & Dinc, E. (2019). A qualitative research on the difficulties and failures about probability concepts

DOI: https://doi.org/10.24127/ajpm.v14i4.14367

of high school students. *Journal of Educational Issues*, *5*(1), 1–19. https://doi.org/10.5296/jei.v5i1.14146

Stalvey, H. E., Burns-Childers, A., Chamberlain, D., Kemp, A., Meadows, L. J., & Vidakovic, D. (2019). Students' Understanding of The Concepts Involved in One-sample Hypothesis Testing. *Journal of Mathematical Behavior*, *53*(March 2019), 42–64. https://doi.org/10.1016/j.jmathb.2018.03.011

Tarr, H., & Maharaj, A. (2021). A Preliminary Genetic Decomposition for Conceptual Understanding of The Indefinite Integral. *Journal of Mathematical Behavior*, *63*(100891), 1–24.

Triliana, T., & Asih, E. C. M. (2019). Analysis of students' errors in solving probability based on Newman's error analysis. *Journal of Physics: Conference Series*, *1211*(1), 012061. https://doi.org/10.1088/1742-6596/1211/1/012061

Widada, W. (2017). Beberapa Dekomposisi Genetik Siswa dalam Memahami Matematika. *Jurnal Pendidikan Matematika Raflesia*, *2*(1), 30–45.

Wijayanti, K. (2017). The Refinement of The Preliminary Genetic Decomposition of Group. *Journal of Physics: Conference Series*, *824*, 012045. https://doi.org/10.1088/1742-6596/755/1/011001