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SECONDARY STUDENTS' COVARIATIONAL REASONING IN SOLVING THE FILLING BOTTLE PROBLEM

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Abstract

Covariational reasoning is closely related to the concept of functions, as both involve relationships between two quantities. Although it is useful for understanding change and interdependence between quantities, the notion of covariation is rarely introduced directly to students since it is not explicitly included in the school curriculum. This study aims to explore students' covariational reasoning in constructing function graphs in the context of a filling bottle problem. Three aspects of covariational reasoning were analyzed: identifying variables, ways of coordinating variables, and quantifying the rate of change. This qualitative research employed a case study design and involved three 11th-grade students selected through purposive sampling. Data were obtained from students' responses to the filling bottle task and interviews, and analyzed through data condensation, data display, and conclusion drawing. The findings indicate that (1) students had difficulty identifying the independent and dependent variables in a functional problem, especially graph, (2) students tended to rely on secondary variables when coordinating the independent and dependent variables, and (3) students' ability to quantify the rate of change depended on their ability to identify and coordinate the two variables.

Keywords: Covariational reasoning; filling bottle problem; functional graph

Abstrak

Pada pelajaran matematika, penalaran kovariasi memiliki kaitan yang erat dengan materi fungsi karena keduanya sama sama memuat hubungan antara dua kuantitas. Meskipun berguna untuk memahami konsep perubahan dan hubungan antar kuantitas, konsep kovariasi jarang dikenalkan secara langsung kepada siswa karena memang tidak tercantum dalam kurikulum pembelajaran di sekolah. Tujuan dari penelitian ini adalah mengeksplorasi penalaran kovariasi siswa dalam mengonstruksi grafik fungsi pada masalah filling bottle. Terdapat tiga aspek penalaran kovariasi yang dianalisis dalam penelitian ini, yaitu mengidentifikasi variabel, cara mengoordinasikan variabel, dan mengkuantifikasi laju perubahan. Penelitian kualitatif dengan jenis studi kasus ini melibatkan tiga siswa kelas XI yang dipilih secara purposive sampling. Data penelitian diperoleh dari hasil lembar tugas filling bottle dan wawancara, kemudian dianalisis menggunakan teknik kondensasi data, penyajian data, dan penarikan kesimpulan. Hasil penelitian menunjukkan bahwa 1) siswa masih kesulitan dalam mengidentifikasi variabel bebas dan variabel terikat dalam suatu masalah fungsi, khususnya grafik, 2) siswa masih bergantung pada variabel perantara dalam mengoordinasikan variabel bebas dan variabel terikat, 3) kemampuan siswa dalam mengkuantifikasi laju perubahan bergantung pada kemampuan mengidentifikasi dan kemampuan mengoordinasikan dua variabel.

***Kata kunci**: Grafik fungsi; masalah filling bottle; penalaran kovariasi*

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INTRODUCTION

Understanding the relationship between two quantities is a fundamental aspect of mathematics learning. This is reflected in the framework of the Programme for International Student Assessment (PISA), in which one of the major topics is change and relationship. The inclusion of this topic emphasizes the importance of understanding change and the interdependence between two quantities. For instance, the relationship between time and the distance traveled by a vehicle, or between the number of goods purchased and the total price to be paid.

In mathematics, such relationships are essential to study because they help students develop knowledge about how one quantity influences another. Moreover, this ability is valuable for explaining phenomena, making predictions, and making appropriate decisions in various real-life contexts (Büchter et al., 2024; Sterner, 2024).

According to Bagossi et al. (2022), in order to model the relationships between two quantities such as the examples above, students need to understand the concept of covariation. Covariation refers to the relationship between two quantities in which a change in one quantity affects the change in the other (Bagossi, 2024; Thompson & Carlson, 2017; Wilkie, 2020). The individual's ability to explain such relationships is referred to as covariational reasoning.

One of the instruments frequently used to investigate covariational reasoning is problems involving dynamic events. Dynamic events are situations in which one variable changes continuously with respect to another variable over time or under certain conditions (Donovan et al., 2024). Among them, the filling bottle problem is one of the most widely used (Johnson et al., 2017). This problem is not only suitable for representing the relationship between two variables but is also familiar in everyday life. In the filling bottle problem, students are asked to observe the process of pouring water into a bottle, identify the relationship between the volume of water and its height, and then construct a graph representing that relationship.

In practice, the concept of covariation is more often encountered in advanced mathematics, particularly calculus. As a result, research on covariational reasoning has primarily been conducted with mathematics undergraduates, pre-service mathematics teachers, or experienced mathematics teachers. For instance, studies by Stalvey & Vidakovic (2015), Kertil et al. (2019), dan Belin dkk. (2024) revealed that even these groups often demonstrate limited understanding of covariation, despite having studied mathematical concepts involving relationships between two variables.

As noted by Bagossi et al. (2022), this issue arises from the limited introduction of covariation concepts at the high school level. In mathematics learning, covariational reasoning is closely related to the topic of functions, as both involve the relationship between two variables (Kertil et al., 2019). However, in school settings, functions are often introduced procedurally, primarily through formulas or function equations. As a result, students tend to fail to understand functions as dynamic relationships between two variables.

For example, in the context of function graphs, students may recognize that the graph of $f(x) = x^2$ is a parabola but do not understand why it takes that shape. Through the concept of covariation, functions are not merely

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about memorizing the shape of a graph from its equation, but rather about emphasizing the patterns of change that occur between the values of x and $f(x)$ when represented graphically.

Based on this background, the present study investigates high school students' covariational reasoning using the filling bottle task. Specifically, the study explores students' covariational reasoning through three aspects developed by Kertil et al. (2019): (1) identifying variable, (2) ways of coordinating variables, and (3) quantifying the rate of change. The aim of this research is to examine how high school students engage in covariational reasoning when solving function graph problems contextualized in the form of a dynamic event. If students are able to demonstrate covariational reasoning across these three aspects, it may provide valuable insights into how the teaching of functions should be approached in schools.

METHODS

This study employed a qualitative approach with a case study design. The task given to students was a modified version of the filling bottle problem developed by Thompson & Carlson (2017), in which three additional questions were included to elicit indicators of students' covariational reasoning in the written test. The task is presented in Figure 1.

The participants were three 11th-grade students from a public senior high school in Jember, East Java, selected through purposive sampling based on several criteria: (1) having completed the topic of function graphs, (2) demonstrating strong mathematical ability, (3) demonstrating good communication skills, (4) being able to construct a graph for the filling bottle problem, and (5) displaying indicators of covariational reasoning in their written responses. Criterion (5) was analyzed based on the extent to which students were able to identify the involved variables (water volume and water height), explain the relationship between the two variables, and show an understanding of the rate of change in the context of the filling bottle graph. The three selected students were subsequently interviewed to confirm their written responses and to further probe their covariational reasoning.

Data analysis followed the framework of Miles et al. (2014) as cited in (Kalpokaite & Radivojevic, 2019) consisting of three stages: data condensation, data display, and conclusion drawing. In the data condensation stage, activities included administering the written test, analyzing students' written responses based on the criteria, and conducting interviews. In the data display stage, the researcher presented students' graphs, interview excerpts, and descriptive accounts of their covariational reasoning. Finally, in the conclusion drawing stage, students' covariational reasoning was interpreted based on the three observed aspects. The indicators for each aspect are presented in Table 1.

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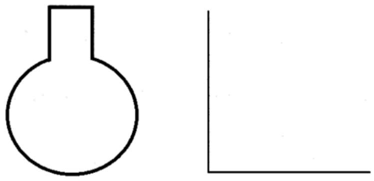
Berikut adalah gambar dari sebuah botol kosong. Jika botol ini diisi air dengan debit konstan hingga penuh, maka gambarkan grafik yang menunjukkan tinggi air sebagai fungsi dari volume air yang masuk!



Apa saja hal yang berubah selama air dituangkan ke dalam botol?
Bagaimana hubungan antara volume air yang dituangkan dengan tinggi air dalam botol?
Apakah perubahan tinggi air selalu sama di setiap bagian botol? Mengapa?

Figure 1. Indonesian version of the filling bottle problem

Table 1. Covariational reasoning indicators

Aspect	Subaspect	Code	Indicator
Identifying Variables	*thinking by primary variables*	IV.PV	The student states that the volume of water is the independent variable and the height of the water is the dependent variable.
	thinking by secondary variables	IV.SV	The student mentions variables other than volume and height of the water.
	thinking by reversing the roles	IV.RV	The student states that the height of the water is the independent variable and the volume of water is the dependent variable.
Ways of coordinating the variables	*uncoordinated way of thinking*	CV.UC	The student is unable to recognize the relationship between volume and height of the water.
	indirect coordination	CV.IC	The student identifies a relationship between volume and height, but only through the width of the bottle.
	direct coordination	CV.DC	The student identifies the relationship between volume and height as always linear.
	direct and systematic coordination	CV.DSC	The student identifies the relationship between volume and height as not always linear.
Quantifying the rate of change	*gross quantification*	QR.GQ	The student observes the changes in volume and height as “fast” or “slow” without providing any clear mathematical justification.
	extensive quantification	QR.EQ	The student focuses only on the changes in water height, assuming that the volume changes at a constant rate.
	intensive quantification	QR.IQ	The student can explain the rate of change between height and volume at each section of the bottle.

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RESULTS AND DISCUSSION

This section presents students' covariational reasoning in solving the filling bottle problem. The researcher is represented by the code R, while the three subjects are represented by the codes S1, S2, and S3.

1. Results of Subject 1 (S1)

The first step taken by S1 in constructing the graph was identifying the variables involved in the filling bottle problem. In the written response, S1 stated water volume and water height as the two changing quantities when water is poured into the bottle. S1 also highlighted the phrase "the height of water as a function of the volume of water" and explained that the function of something is the output of that quantity. Accordingly, S1 chose volume as the input on the *x*-axis and water height as the output on the *y*-axis. The following excerpt illustrates S1's reasoning in identifying the variables.

R : *What do you understand from the phrase "the height of water as a function of the volume of water"?*
S1 : *From that phrase, it means "the height of water as the output of the volume of water." Therefore, x is the water volume and y is the water height.*
R : *So, in your opinion, which quantity influences the other?*
S1 : *The water volume influences the water height.* (IV.PV)

Based on the excerpt, it can be seen that S1 reasoned using primary variables. In other words, S1 was able to correctly identify the independent and dependent variables.

The next step taken by S1 to determine the relationship between water volume and water height was observing the difference in the amount of water needed in each section of the bottle. S1 stated that the difference in water requirements between the narrow and wide sections of the bottle affects the rate at which the water height increases. According to S1, when water fills a wider section, the height continues to rise but at a slower rate. Conversely, when water fills a narrower section, the water height rises more quickly. The following excerpt shows how S1 coordinated the variables.

R : *How would you describe the relationship between water volume and water height?*
S1 : *The more water is poured, the higher the water level becomes.*
R : *Does the increase in water height remain the same in every section of the bottle?*
S1 : *Not always. When the water fills the wider part, the increase in height is slower compared to when it fills the narrower part.* (CV.DSC)
R : *In which part does the height take the longest to rise?*
S1 : *In the middle part of the bottle, which is circular in shape.*

S1 also realized that differences in the increase of water height would result in variations in the slope of the graph. Therefore, S1 shaded certain sections of the bottle to indicate changes in the rate at which the water height increased. The partitioned sections of the bottle and S1's constructed graph are presented in Figure 2.

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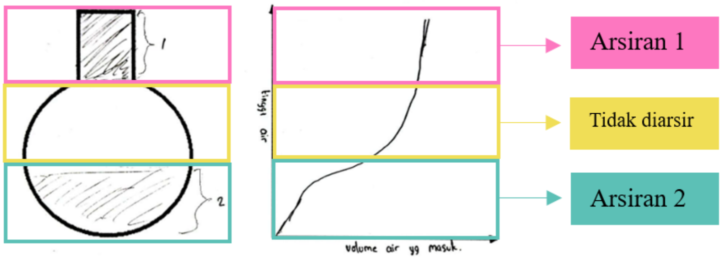

Figure 2. S1's graph result

S1's observation of the varying increases in water height across different parts of the bottle indicates that S1 was able to coordinate water volume and water height directly and systematically. The researcher then further explored how S1 constructed the graph by asking S1 to explain the reasoning behind determining the slope of the constructed graph. This was intended to confirm whether S1 understood the rate of change between the two variables. The following excerpt illustrates S1's reasoning in quantifying the rate of change.

R : Why did you draw the graph in this way?

S1 : I just estimated it. For example, in Shading 2, the graph slopes down a little, then as the water approaches the middle of the circular part, it slows down, so the slope becomes flatter, and then it becomes steeper again as it reaches the end of the circle. So, after being flat, it rises steeply again. In Shading 1, it is a straight line.

R : Suppose we take one point in the steep section and another point in the flat section. What might their coordinates be?

S1 : For example, the first point has a volume of 1 liter and a height of 10 cm. The second point, after adding another liter (so 2 liters), the height increases by around 15 cm. So, from 0 liters to 1 liter, the water height rises 10 cm, while from 1 liter to 2 liters, it only rises 5 cm. In Shading 1, if 1 liter raises it by 10 cm, then 2 liters would be 10×2 and 3 liters would be 30 cm, so it is constant. (QR.IQ)

Based on this excerpt, it can be seen that S1 was able to explain the change in the ratio of water height to water volume in different parts of the bottle by providing examples of coordinate points on the Cartesian plane. This shows that S1 constructed the graph by considering the changes in water volume and height along the x and y axes. In other words, S1 demonstrated the aspect of intensive quantification.

2. Results of Subject 2 (S2)

S2 was able to identify that the two main variables under observation in this case were the volume of water and the height of water. However, S2 did not fully understand the meaning of the statement "height as a function of volume." According to S2, since the problem stated that height was the function, height should be placed on the x-axis. This was the opposite of S2's written answer. Yet, after answering several interview questions, S2 eventually decided to place the volume of water on the x-axis and the height of water on the y-axis. The following excerpt illustrates how S2 identified the variables.

R : What do you understand from the statement "the height of

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water as a function of the volume of water"?

S2 : *The height of water acts as x, because it says "height of water as a function."*

R : *But why does your answer show the opposite?*

S2 : *I thought the volume was the function, so I placed it on the x-axis. Because usually, in problems I have encountered, volume is always on the x-axis. But this problem states that height is the function. So, I thought height should be on the x-axis. But, if I think again, both seem correct. The volume of water can affect the height, but the height of water can also affect the volume.*

R : *Then why did you finally decide to put the volume of water on the x-axis?*

S2 : *Because the volume of water should influence the height of water. The height adjusts to the amount of water entering the different spaces of the bottle.*

From this excerpt, it can be seen that S2 initially thought in reverse before reconsidering through the notion of primary variables. S2 appeared to struggle in understanding the meaning of the statement in the problem, since the decision to place volume on the x-axis was mainly based on prior experience with similar tasks. S2 did not truly grasp the role of the independent and dependent variables intended in the problem.

Next, S2 attempted to determine the relationship between the two variables by focusing on two aspects: the diameter of the bottle and the increase in height. S2 stated that the larger the diameter of the bottle, the slower the height of water would increase. Conversely, the smaller the diameter, the faster the height of water would rise. The following interview excerpt shows how S2 coordinated the two variables.

R : *How do you see the relationship between the volume of water and the height of water?*

S2 : *The more water volume in the bottle, the higher the water level.*

R : *Does the height always increase at the same rate in every part of the bottle?*

S2 : *No, it depends on the size. Since the bottle is circular, some spaces are larger and others are smaller, so the increase in height is not constant. Sometimes it increases more quickly, sometimes more slowly. When it increases more slowly, the height does not rise as much as before, even though the added volume is the same as before.* (CV.DSC)

S2 also realized that the varying diameter of the bottle caused the water height not to increase at a constant rate. According to S2, such changes in the rate of increase would make the graph non-linear. To illustrate more clearly which parts of the bottle affected these changes, S2 divided the bottle into three sections. The division of the bottle and S2's graph construction can be seen in Figure 3.

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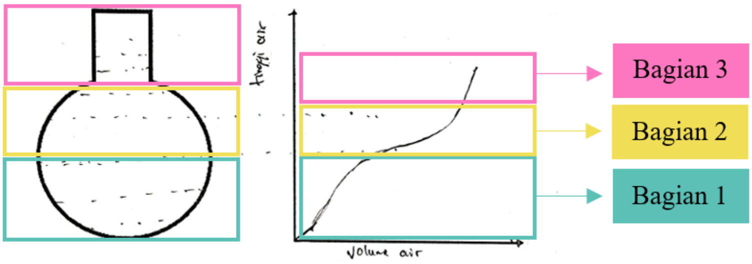

Figure 3. S2's graph result

Similar to S1, S2 also focused on the varying increases in water height in each section of the bottle. This indicates that S2 was also able to coordinate the relationship between water volume and water height in a direct and systematic way.

Next, the researcher explored further how S2 constructed the graph and asked S2 to explain the way he determined the slope of the graph he had drawn. The following excerpt illustrates how S2 quantified the rate of change.

R : Why does your graph look like this?

S2 : In Part 1, the shape of the bottle widens, which makes the water height rise slowly, so the graph curves downward. In Part 2, it goes from wide to narrow. The water, which initially rose slowly, then rises faster, so the graph becomes gentle and then steep. In Part 3, the rate of increase is constant, so the graph is linear.

R : Is the increase the same in every part of the bottle?

S2 : No, but the speed changes. In Part 1, the speed starts to slow down. For example, at first it was 1, then it slowed down to 0.8, 0.7. In Part 2, it's the opposite, from slow to fast, so from 0.7, 0.8, then 1.

R : Did you estimate the coordinates when drawing this graph?

S2 : No.

R : So, if the water rises slowly, the graph is gentle, and if the water rises quickly, the graph is steep, is that right?

S2 : Yes, I only drew it based on how fast the water level increased.

Based on this interview excerpt, it can be seen that S2 did not draw the graph based on the relationship between the numerical values of volume and water height in the Cartesian plane. Instead, S2 perceived the slope of the graph merely as a representation of how fast or slow the water height increased, without providing clear mathematical calculations. This indicates that S2 only demonstrated the aspect of gross quantification.

3. Results of Subject 3 (S3)

S3 stated that the two main focuses in the filling bottle problem were the volume of water and the rate of increase in water height. S3 wrote the volume of water on the y-axis and the rate of increase in water height on the x-axis. This indicates that S3 had not yet understood the meaning of the statement "height as a function of volume."

S3 argue that, the wider the bottle, the slower the water would rise to the surface. In this case, S3 tended to observe only the rate of increase in height for each volume added into the bottle, rather than directly examining the relationship between water volume and water height. The following is an excerpt

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from the interview with S3 in identifying variables:

R : *What do you understand from the statement "water height as a function of water volume"?*
S3 : *If the volume of water increases, then the rate of increase in water height is what is calculated.* (IV.SV)
R : *Why the rate of increase in water height?*
S3 : *Because the wider the bottle, the slower the water height increases. But if the bottle gets narrower, the rate of increase in height becomes faster, or it fills more quickly.*
R : *Why did you put the rate of increase in water height on the x-axis and the volume on the y-axis?*
S3 : *Because the volume of water is what is poured in, so the illustration is more upward, while the rate is more to the right. So, the relationship is one to one. It could also be reversed.*

Based on the interview excerpt, it can be seen that S3 reasoned through a secondary variable by introducing another quantity, namely the rate of increase in water height. Similar to S2, S3 did not fully understand the roles of the independent and dependent variables intended in the problem.

Afterward, in terms of coordinating variables, S3 mentioned the cross-sectional area of the bottle as another factor affecting the change in water height. According to S3, the cross-sectional area significantly influenced the relationship between the rate of increase in height and the volume of water. This suggests that S3 was able to perceive how fast the height of water increased, but could not clearly articulate the role of volume and its relationship to water height. This indicates that S3 demonstrated indirect coordination, namely seeing the relationship between water volume and water height through another variable. The following excerpt illustrates S3's reasoning in coordinating variables:

R : *What is the relationship between water volume and water height?*
S3 : *The wider the bottle, the larger the volume, and the longer it takes to fill.*
R : *So, in this case, which quantity influences the other?*
S3 : *The influence is more on the area, the cross-sectional area of the bottle affects the rate of filling. So, the wider the bottle, the longer it takes to fill.* (CV.IC)
R : *Does the change in water height always remain the same in every part of the bottle?*
S3 : *Not always, because the sides of the bottle are not constant. The part of the bottle with the largest cross-sectional area makes the height rise more slowly.*

Furthermore, the researcher examined this aspect more closely by exploring S3's reasoning in constructing a solution strategy. The researcher sought to find out how S3 could construct a graph without directly connecting the main variables. It turned out that S3 used a strategy not much different from S1 and S2. S3 divided the bottle into five segments that indicated changes in the rate of increase in water height and reflected differences in the steepness of the graph. The segmentation of the bottle and the resulting graph constructed by S3 can be seen in Figure 4.

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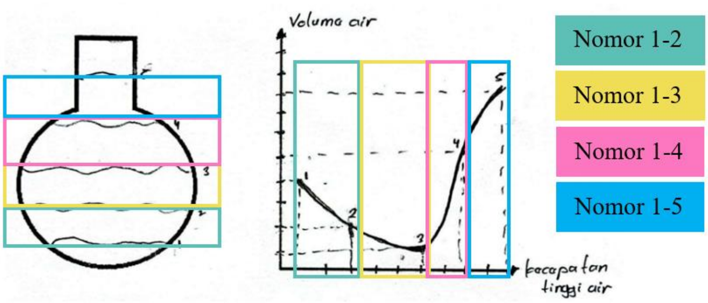

Figure 4. S3's graph result

During the interview, the researcher explored further how S3 constructed the graph and asked S3 to explain how the slope of the constructed graph was determined. The following excerpt shows S3's explanation in quantifying the rate of change.

R : *Why did you start the initial point of the graph from here?*

S3 : *So, this steep part means that the filling speed slows down, that's why the graph goes down.*

R : *Which part of the bottle does this downward curve represent?*

S3 : *That is from point 1 to point 3.*

R : *Then what does point 3 to point 5 mean? Why is the slope like this?*

S3 : *Because the tube is narrowing, it becomes steeper upwards. The filling speed is getting faster.*

After several questions, S3 realized that there was a mistake in placing the variables on the axes. According to S3, the *x*-axis should represent the volume of water, while the *y*-axis should represent the filling speed of the water. The following excerpt shows this clarification.

R : *From the base to the middle of the bottle, the rise of the water slows down. Why does your graph move to the right? If it moves further to the right, the x-axis values increase, which means the speed of the water height should be faster, not slower, right?*

S3 : *Oh, I think that was reversed. The filling speed should be on the y-axis, and the volume should be on the x-axis. So, on the x-axis there are five points, each representing 20%, 40%, 60%, 80%, and 100%. On the y-axis, the values are in units, starting from 1, 2, 3, 4, 5, and so on.*

Based on the interview excerpt, it can be seen that S3 interpreted the volume of water (on the *x*-axis) as values expressed in percentages, while the filling speed of water height (on the *y*-axis) was expressed in units. The revised graph of S3 can be seen in Figure 5.

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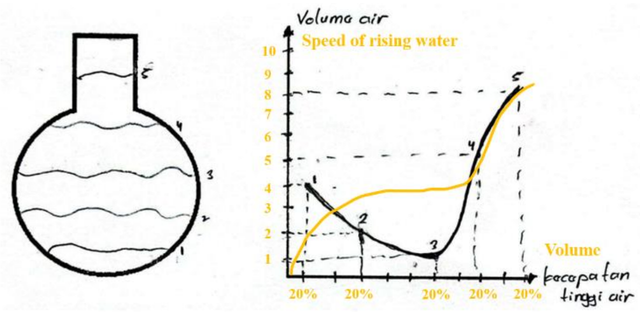

Figure 5. S3's graph revised

This revision did not change S3's reasoning process, since he still understood the graph only in terms of faster or slower increases in the water height inside the bottle. S3 was not yet able to understand how the change in water height occurs as a consequence of changes in water volume. In other words, S3 was still operating at the level of gross quantification.

For further clarity, the summary of the covariational reasoning of the three subjects is presented in Table 2.

Table 2. Summary of subject's covariational reasoning

Students' Covariational Reasoning		
S1	S2	S3
1. Thinking by primary variables 2. Direct and systematic coordination 3. Intensive quantification	1. Thinking by primary variables 2. Direct and systematic coordination 3. Gross quantification	1. Thinking by secondary variables 2. Indirect coordination 3. Gross quantification

Based on the data analysis, the following section discusses students' covariational reasoning in solving the filling bottle problem.

1. Identifying Variables

The researcher found that students still struggled to understand the statement "the height of water as a function of the volume of water." Although it seems simple, this statement requires a conceptual understanding of function. Moleko (2021) argues that difficulties in understanding word problems may stem from students' limited mathematical vocabulary. This aligns with the classroom findings, which show that the concept of function is more often introduced symbolically, namely in the form of formulas or function equations.

According to Dinkelman & Cavey (2015), conceptual understanding of function has not yet been widely mastered by students. Graph problems presented only in the form of function formulas tend to make students think statically (Moore & Thompson, 2015). In other words, students cannot recognize the role of each variable involved in a function problem and instead view graphs merely as fixed visual representations. For example, students may simply memorize that the graph of a quadratic function is a parabola without knowing why this is the case.

Furthermore, the researcher found that some students identified other

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variables in addition to water volume and water height. This way of thinking was influenced by the tendency of students to focus on the most easily observable variables, in this case "the cross-sectional area" of the bottle. This finding is consistent with previous studies that also reported the emergence of other variables such as "time" in graph construction. For instance, the study of Paoletti et al. (2022), Stalvey & Vidakovic (2015), Yemen-Karpuzcu et al. (2017).

In the study of Kertil et al. (2019), one of the subjects even stated that adding the time variable had become a common habit when constructing graphs. This suggests that students are accustomed to selecting variables they frequently encounter in their prior learning experiences.

2. Ways of Coordinating Variables

The researcher found that some students experienced difficulties in directly understanding the relationship between the volume of water and its height. This difficulty arises because these two variables are not easily observable. Johnson et al. (2017) pointed out that, when dealing with covariational problems, students tend to choose variables that are easier to measure or calculate.

In this study, another variable that emerged was "the cross-sectional area" of the bottle. According to the students, the cross-sectional area plays an important role in determining how fast or slow the water level rises. This indicates a reasoning pattern within the sub-aspect of indirect coordination. In other words, students considered that the change in either volume or water height must be mediated through the cross-sectional area.

The inclusion of another variable to understand the relationship between the primary variables reflects a way of thinking through secondary variables. Menurut Kertil (2020), emphasized that introducing additional variables is not an absolute mistake. However, such reasoning may hinder the development of covariational reasoning. Consequently, students may find it difficult to perceive the direct relationship between two variables in a functional problem. Moreover, they might fail to recognize nonlinear situations in functional problems because they do not grasp the concept of the rate of change between the two primary variables.

This challenge is also evident among students who perceive the roles of the primary variables in reverse. If such reasoning is not addressed, students may struggle to identify which variable is independent and which is dependent. As a result, they may misinterpret the graph. In addition, reasoning in reverse could also hinder the understanding of other mathematical concepts, such as derivatives, slope of graphs, and functional modeling.

3. Quantifying the Rate of Change

Quantifying the rate of change refers to observing how the pattern of change in the water height occurs as a result of changes in the water volume. This aspect is a new framework developed by Kertil to investigate the extent to which students can understand the shape of the graphs they construct. The reason is that an understanding of graphs, such as slope and steepness, needs to be analyzed through students' ability to quantify the rate of change between two variables. In other words, Kertil aims to explore how students comprehend the rate at which one variable changes as a

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result of the other, then visualize this in the form of a graph.

In this study, two different perspectives emerged in constructing and determining the slope of the graph. S1 constructed the graph based on the changes in water volume and height, represented by coordinate points on the x-axis and y-axis. Menurut Moore & Thompson (2015), this is referred to as emergent shape thinking, namely the ability to understand the shape of a graph as a relationship between water volume and water height that continuously changes. Johnson et al. (2020) similarly described S1's behavior as a covariational representation, which is the ability to represent a graph as the relationship between two quantities that vary simultaneously.

Meanwhile, S2 and S3 constructed their graphs based on the perceived speed of the rising water height in each section of the bottle. According to Patterson & McGraw (2018), this behavior can be described as chronicle-based thinking, namely viewing the shape of the graph as a journey of rising or falling values of a variable.

CONCLUSION AND SUGGESTION

Based on the discussion of the three aspects of covariatinal reasoning described earlier, it can be concluded that students' covariational reasoning in constructing function graphs in the filling bottle task is an important topic to be further investigated and developed. The ways in which students reason and justify their arguments through the three aspects of covariational reasoning indicate the necessity of fostering this ability, particularly at the senior high school level.

Although the concept of covariation is not explicitly included in the mathematics curriculum at this level, teachers should begin introducing it before students learn about functions and their graphs. This introduction will help students perceive graphs not merely as static objects but as representations of the roles of variables, the relationships between them, and the rate of change involved.

Based on the analysis of written work and interviews, the researcher believes that the concept of covariation can be implemented in high school mathematics instruction. Therefore, the following suggestions are proposed: 1) There needs to be an introduction of covariation concepts before students study functions and function graphs, 2) The teaching of functions should not only emphasize symbolic problems but also incorporate a variety of contextual problems, and 3) Teachers should provide more practice in solving function graph problems that encourage the development of students' covariational reasoning.

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