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INVESTIGATING PRE-SERVICE TEACHERS' (PST) MATHEMATICAL THINKING TRAJECTORIES (MTT) IN SOLVING SYSTEM OF LINEAR EQUATIONS PROBLEMS

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Abstract

Linear algebra is an area of mathematics that is widely considered challenging. One of the subjects covered is the system of linear equations in two variables. During their first year of college, Pre-Service Teachers (PST) study this subject using increasingly sophisticated approaches and theories. The objective of this qualitative study is to classify and describe the Mathematical Thinking Trajectories (MTT) of PST in solving a system of linear equations in two variables problems. Thirty-three PST majoring in mathematics education who enrolled in the Errors and Misconceptions in Mathematics course participated in this study. Data collected using tests and interviews, and analyzed using triangulation method. The Three Worlds of Mathematics (TWM) theory framework is used in this study. Seven MTT classifications were identified by the research findings: (1) embodied, (2) symbolic, (3) embodied-symbolic, (4) symbolic-embodied, (5) formal, (6) embodied-symbolic-embodied, and (7) symbolic-embodied-symbolic. "Embodied" refers to how PST applies mathematical ideas that focus on actual objects, "symbolic" focuses on symbols, operations, and their properties, and "formal" focuses on definitions, theorems, and logical procedures. In learning practices, the results of this study can be applied to create learning that is meaningful and that fosters the growth of algebraic thinking abilities.

Keywords: Algebraic thinking; mathematical thinking trajectory; meaningful learning; system of linear equations; three worlds of mathematics theory.

Abstrak

Aljabar linear diakui secara luas sebagai salah satu cabang matematika yang dianggap sulit. Salah satu topik yang dipelajari adalah sistem persamaan linear dua variabel. Pada tahun pertama tingkat universitas, mahasiswa belajar topik ini dengan pendekatan dan teori yang lebih kompleks. Penelitian ini adalah penelitian kualitatif yang bertujuan untuk mengklasifikasikan dan mendeskripsikan lintasan berpikir matematis (LBM) mahasiswa dalam menyelesaikan masalah sistem persamaan linear dua variabel. Partisipan yang terlibat ada 33 mahasiswa program sarjana pendidikan matematika yang menempuh mata kuliah Kesalahan dan Miskonsepsi dalam Matematika. Pengumpulan data melalui tes dan wawancara, dan dianalisis menggunakan metode triangulasi. Penelitian ini menggunakan kerangka kerja teori Tiga Dunia Matematika (TDM). Hasil penelitian menemukan 7 klasifikasi LBM, yaitu (1) embodied, (2) simbolik, (3) embodied-simbolik, (4) simbolik-embodied, (5) formal, (6) embodied-simbolik-embodied, dan (7) simbolik-embodied-simbolik. Diskripsi 'embodied' mengacu ke penggunaan ide-ide matematis oleh mahasiswa yang fokus ke objek yang sifatnya fisik, 'simbolik' fokus ke simbol, operasi, dan sifat-sifatnya, sedangkan 'formal' fokus pada definisi, teorema, dan langkah-langkah logis. Dalam praktik pembelajaran, temuan penelitian ini dapat digunakan untuk merancang pembelajaran yang bermakna dan juga yang mengembangkan keterampilan berpikir aljabaris.

Kata kunci: *Berpikir Aljabaris; lintasan berpikir matematis; pembelajaran bermakna; sistem persamaan linear; teori tiga dunia matematika.*

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INTRODUCTION

Pre-service teachers (PST) bring a variety of prior knowledge to the university. This diversity gives a chance to identify variations in the ways PST chooses to engage with knowledge (Alreshidi, 2023; Hailikari, 2009; Thurn et al., 2022), including mathematical knowledge. Diverse prior knowledge results in heterogeneity and becomes a challenge in learning (Hailikari, 2009; Kosiol et al., 2019). Additionally, the quality of learning and learning outcomes may be impacted by this diversity (Hailikari, 2009; Thurn et al., 2022; Witherby et al., 2023). There are at least two impacts of prior knowledge in learning: (1) the direct influence of prior knowledge in facilitating learning and (2) the influence of the qualities inherent in prior knowledge, for example, misconceptions and the incomplete structure of prior knowledge itself (Hailikari, 2009; Lee et al., 2019).

Prior knowledge is an enormous scope of research study (Hailikari, 2009**).** Prior knowledge is defined in a variety of ways. According to (Hailikari, 2009; Lee et al., 2019)**,** prior knowledge is knowledge that (1) consists of declarative knowledge and procedural knowledge (Alreshidi, 2023), (2) exists prior to task implementation, (3) is available, can be remembered, and can be reconstructed, and (4) can be applied to learning tasks (Binder et al., 2019; Song et al., 2016). Prior knowledge is an inventory in the long-term memory of an individual at their initial point of learning (Thurn et al., 2022), including information, skills, experiences, beliefs, and memories (Alreshidi, 2023; Grogan et al., 2023). On the other hand, prior knowledge is defined by Heitzmann et al. (2023) and Greve et al. (2019) as knowledge that is stored in the memory of an individual and can be utilized to accomplish tasks.

Prior knowledge has been the subject of numerous studies. For instance, Hailikari (2009) investigates how various prior knowledge types affect student achievement as well as how different methods of assessment affect the observed effects of prior knowledge. Thurn et al. (2022) investigated how prior knowledge mediated the relationship between intelligence and proportional reasoning learning. Alreshidi (2023) investigated how improving students' prior knowledge of particular mathematical topics affects their achievement. Witherby et al. (2023) investigated the connection between prior knowledge and student prediction accuracy; the findings indicated a negative relationship between the two.

This article reports the results of research that investigated how PST use their prior knowledge in solving mathematical problems. The definition of prior knowledge used in this research refers to (Hailikari, 2009; Heitzmann et al., 2023; Lee et al., 2019). The form of prior knowledge in this research is in the form of mathematical ideas stored in PST's long-term memory, which are used to solve the system of linear equations in two variables problems. In this study, the sequence in which PST applies mathematical ideas is called the Mathematical Thinking Trajectory (MTT).

This research uses the Three Worlds of Mathematics (TWM) theoretical framework. This theory examines a long-term mathematical thinking framework (Tall, 2019, 2020b). The idea of the emergence of this theory is based on three stages of development of mathematical thinking which are increasingly sophisticated and

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interconnected. According to Chin et al. (2022) and (Tall, 2008, 2020a), the three stages of development of mathematical thinking are (1) conceptual embodiment, (2) operational symbolism, and (3) axiomatic formalism, which will be abbreviated to (1) embodied, (2) symbolic, and (3) formal.

This TWM theory can be applied to specific mathematical topics, even though its ideas are long-term in nature (Kidron & Tall, 2015; Tall & Healy, 2014; Tall & Katz, 2014)**.** Tall et al. (2014) apply TWM theory to observe how students' understanding shifts from linear to quadratic equations. The students did not exhibit the "didactic cut" of (Filloy & Rojano (1989), but they used 'procedural embodiments', shifting terms around with added "rules" to obtain the correct answer (De Lima & Tall, 2008). Kidron & Tall (2015) investigate how the approach using Mathematica software to visualize the convergence of a sequence of functions as a sequence of objects (graphics) to a fixed object (the graph of the limit function) can facilitate a flexible combination of symbolic and visual representations that could potentially lead to a formal definition of the concept of limits. Tall & Katz (2014) investigate the development of MTT from human perception and action to increasingly sophisticated forms of reasoning and proof in terms of Cauchy's ideas of function, continuity, limits, and infinitesimals using theoretical frameworks from mathematics education and cognitive psychology. However, the shifts within three stages based on TWM framework in terms of students' MTT still need to be studied.

As a result, keeping with the use of the research framework previously mentioned, this study uses the TWM theoretical framework with the aim to classify and describe PST' MTT in solving certain topic problems in linear algebra, i.e., the system of linear equations in two variables. The results of this study are expected to complement research with a similar framework, i.e., TWM, but with a new perspective, i.e., MTT.

METHODS

This research is descriptive qualitative research. The purpose of this study is to classify and describe PST's MTT. MTT is a sequence of PST's use of mathematical ideas in solving a system of linear equations in two-variable problems. The participants were 33 PST from one class majoring in mathematics education who enrolled in the Errors and Misconceptions in Mathematics course. The selection of this class because there were still errors and misconceptions in the system of linear equations in two-variable topics during the class, although they have completed prerequisite courses (i.e., Basic Mathematics, Logic and Mathematical Proof, and Elementary Linear Algebra), thus, in theory, PST already possesses the prior knowledge needed. The instruments used to explore data are test and interview. A week before the test, PST received a review on the subject of systems of linear equations. Three PST as a group presented the topic and carried on the discussion in class. The time provided to solve the problems in the test is 20 minutes. The test is given classically. The problems in the test are shown in Figure 1.

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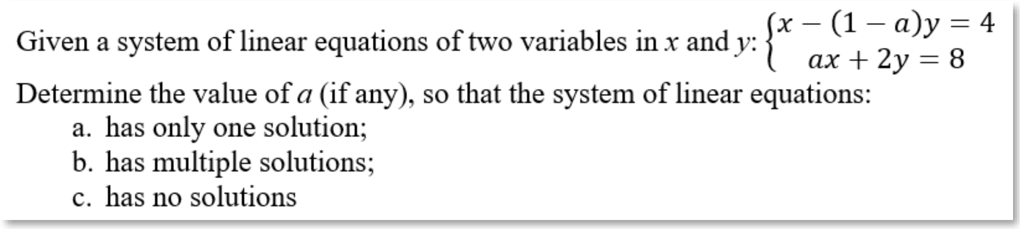

Given a system of linear equations of two variables in x and y: $\begin{cases} x-(1-a)y=4 \\ ax+2y=8 \end{cases}$

Determine the value of a (if any), so that the system of linear equations:

a. has only one solution;
b. has multiple solutions;
c. has no solutions

Figure 1. Research Instrument

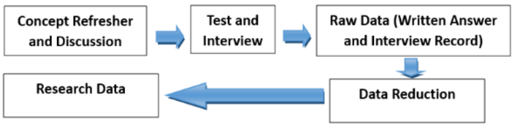

Figure 2. Data Collection Procedure

Table 1. Using the TWM Theory in The Context of Solving System of Linear Equations in Two Variables Problems.

The Three Worlds of Mathematics	Focus	The Example of Mathematical Ideas
Embodied	On the physical object of each equation in the system	1. The graph of two equations is straight lines; 2. The graph of two equations is two parallel lines; 3. The graph of two equations is two coincident lines; 4. The graphs of two equations intersect; 5. Drawing graphs.
Symbolic	On symbols, operations, and their properties	1. Multiplying equation by a number; 2. Adding or subtracting equations; 3. Substituting a number into an equation; 4. $-\frac{a_1}{b_1}$ is the gradient of the equation; 5. If $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, then the system has many solutions; 6. If $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$, then the system has no solution.
Formal	On definitions, theorems, and logical steps	1. If the determinant of the system's coefficient matrix is not equal to zero, then the system has only one solution; 2. If the determinant of the system's coefficient matrix is equal to zero, then the system has many solutions or has no solutions;

The data collected are PST written answers and interview records. Data reduction was then applied to the raw data. It aims to improve the manageability and interpretability of complex data. This process involves some steps of simplifying, grouping, and reducing data without losing any of its substance. Data reduction is also implemented to improve understanding of research findings. The research data collection procedures are shown in Figure 2.

Data were analyzed using the triangulation method. Triangulation is the process of strengthening data from various types of sources. The study's supporting evidence is strengthened by examining the accuracy of the findings and interpreting the data from the same source with different ways, i.e., through written answers and interview records. Data reduction, data presentation, and conclusion drawing were all steps in the triangulation process.

The framework used is the TWM theory. This theory includes three worlds, i.e., embodied, symbolic, and formal (Chin et al., 2022; Tall, 2008, 2020a). In the context of a system of linear equations in two variables, the "embodied" world refers to the use of mathematical ideas that focus on the physical object of each equation in the system. The "symbolic" world refers to the use of mathematical ideas that focus on symbols, operations, and their properties. Meanwhile, the "formal" world refers to the use of mathematical ideas that focus on definitions, theorems, and logical steps. The mathematical ideas of the 3 worlds above are presented in Table 1.

According to TWM theory, the three realms are not independent. PSTs may change between two worlds in the context of MTT. Tall has shown this movement, as seen in Figure 3.

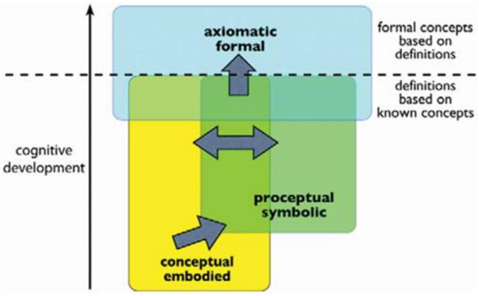

Figure 3. Shifts between Three Worlds of Mathematics

RESULTS AND DISCUSSION

Thirty-three PST' answers served as the source of data for this study. The answers were classified based on their MTT. Seven MTT classifications are identified: (1) embodied, (2) symbolic, (3) embodied-symbolic, (4) symbolic-embodied; (5) formal; (6) embodied-symbolic-embodied; and (7) symbolic-embodied-symbolic. Details regarding the MTT classification of 33 PST in this study are shown in Figure 4.

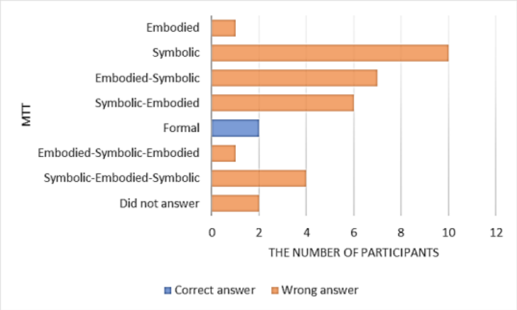

Figure 4. MTT Classification of 33 PST in Solving System of Linear Equations in Two Variables Problems

Figure 4 indicates that the majority of PST MTTs (30.3%) are symbolic, with 10 PST in this category. Embodied-symbolic is on the second-ranked with 7 PST (21.2%), followed by the symbolic-embodied amount of 6 PST (18.2%), symbolic-embodied-symbolic as many as 4 PST (12.1%), formal with 2 PST (6.1%), and both embodied and embodied-symbolic-embodied equivalent to 1 PST (3%). Meanwhile, there were 2 PST (6.1%) who did not answer. Additionally, Figure 4 shows that none of the PST answers with (1) embodied, (2) symbolic, (3) embodied-symbolic, (4) symbolic-embodied, (5) embodied-symbolic-embodied, and (6) symbolic-embodied-symbolic MTT are correct. In contrast, all of the PST answers with formal MTT are correct. The answers are considered to be correct

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if all of the questions in parts a, b, and c in the instrument are correctly answered. If there is at least one incorrect answer, then the answer is considered a wrong answer. Therefore, Figure 4 shows that, out of the 33 answers, two PSTs answered correctly, and the remaining 31 had wrong answers.

Following additional examination, two categories emerged from the 10 PST with symbolic MTT in terms of how they applied mathematical ideas. The sequence in which mathematical ideas are applied to solve problems determines the distinctions between these categories. Additionally, one category emerged from the MTT: embodied, embodied-symbolic, symbolic-embodied, formal, embodied-symbolic-embodied, and symbolic-embodied-symbolic. Table 2 below shows the MTT classification, the number of categories, and the sequence in which PSTs apply mathematical ideas.

Table 2. MTT Classification, the Number of Categories, and the Sequence of PST's Mathematical Ideas in Solving System of Linear Equations in Two Variables Problems

No	**MTT**	**The number of categories**	**The sequence of PST' mathematical ideas**	
1.	Embodied	1	1.	1.1. The system has only one solution if the graphs intersect; 1.2. The system has many solutions if the graphs coincide; 1.3. The system has no solution if the graphs are parallel.
2.	Symbolic	2	1.	1.1. The system has only one solution: if $\frac{1}{a} \neq \frac{-(1-a)}{2}$; 1.2. The system has many solutions: if $\frac{1}{a} = \frac{-(1-a)}{2} = \frac{4}{8}$; 1.3. The system has no solution: if $\frac{1}{a} = \frac{-(1-a)}{2} \neq \frac{4}{8}$; 1.4. Determining the number a for the three conditions above.
			2.	2.1. Multiplying the first equation by the number a; 2.2. Subtracting the first equation with the second equation until getting $y = \frac{4a-8}{a^2-a-2}$; 2.3. The system has no solution if y is undefined; 2.4. The system has many solutions if y is defined; 2.5. Determining the number a for the two conditions above.
3.	Embodied-Symbolic	1	1.	1.1. The system has one solution if the graphs intersect at one point;

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No	MTT	The number of categories	The sequence of PST' mathematical ideas
			1.2.The system has many solutions if the graphs coincide; 1.3.The system has no solution if the graphs are parallel; 1.4.Expressing y in terms of x in both equations; 1.5.Determining the number a for the three conditions above.
4.	Symbolic-Embodied	1	1. 1.1.Expressing y as a function of x in the second equation; 1.2.Substituting y into the first equation; 1.3.The system has one solution if there is no definite value of a; 1.4.The system has many solutions if the graphs coincide; 1.5.The system has no solution if the graphs are parallel
5.	Formal	1	1. 1.1.Expressing the system in the form of a matrix equation; 1.2.Calculating the determinant of the coefficient matrix; 1.3.Determining the inverse of the coefficient matrix; 1.4.The system has no solution if the determinant is zero; 1.5.The system has one solution if the determinant is not zero; 1.6.The system has many solutions if the determinant is zero; 1.7.Determining the number a and checking it to the system for the three conditions above.
6.	Embodied-Symbolic-Embodied	1	1. 1.1.The system has no solution if the graphs are parallel and do not intersect; 1.2.The system has one solution if the graphs intersect; 1.3.The system has many solutions if the graphs coincide; 1.4.Expressing y in terms of x in both equations; 1.5.Determining the number a for the three conditions above; 1.6.Drawing the graphs.

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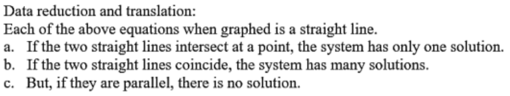
No	**MTT**	**The number of categories**	**The sequence of PST' mathematical ideas**
7.	Symbolic–Embodied–Symbolic	1	1. 1.1. Determining the number a; 1.2. The system has one solution if the graphs intersect; 1.3. The system has many solutions if the graphs coincide; 1.4. The system has no solution if the graphs are parallel; 1.5. Expressing y in terms of x in both equation

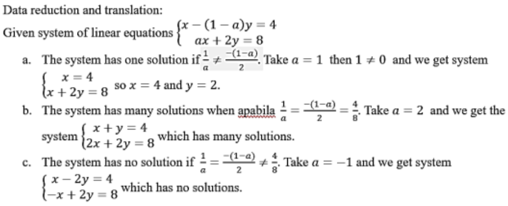
The first MTT is the "embodied" MTT. According to Table 2, there is only one category for this MTT. PST answers in this category of MTT focus on the physical form of the equations in the system, such as stating that the graphs of the equations are two straight lines, then determining the relative position of the two graphs, but cannot determine the value of "*a*" asked on the instrument. PST answer in "embodied" MTT can be seen in Figure 5.



Data reduction and translation:
Each of the above equations when graphed is a straight line.
a. If the two straight lines intersect at a point, the system has only one solution.
b. If the two straight lines coincide, the system has many solutions.
c. But, if they are parallel, there is no solution.

Figure 5. PST Answer in the "embodied" MTT

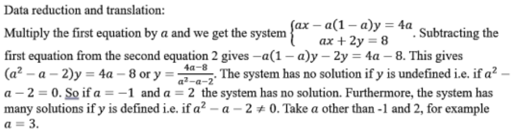
The second is a "symbolic" MTT. According to Table 2, there are 2 categories for this MTT. PST answers in these categories are focused on symbols, operations, and their properties. Answers that are classified into category 1 show that PST concentrates first on the coefficients of the variables *x* and *y* in the two equations, as well as the constants. Then, they compare the corresponding coefficients and constants and determine the value of "*a*". PST answer in the "symbolic" MTT category 1 can be seen in Figure 6.



Data reduction and translation:
Given system of linear equations $\begin{cases} x-(1-a)y=4 \\ ax+2y=8 \end{cases}$
a. The system has one solution if $\frac{1}{a} \neq \frac{-(1-a)}{2}$. Take $a=1$ then $1 \neq 0$ and we get system $\begin{cases} x=4 \\ x+2y=8 \end{cases}$ so $x=4$ and $y=2$.
b. The system has many solutions when apabila $\frac{1}{a}=\frac{-(1-a)}{2}=\frac{4}{8}$. Take $a=2$ and we get the system $\begin{cases} x+y=4 \\ 2x+2y=8 \end{cases}$ which has many solutions.
c. The system has no solution if $\frac{1}{a}=\frac{-(1-a)}{2} \neq \frac{4}{8}$. Take $a=-1$ and we get system $\begin{cases} x-2y=4 \\ -x+2y=8 \end{cases}$ which has no solutions.

Figure 6. PST Answer in the "symbolic" MTT Category 1

Answers that are classified into category 2 show that PST first multiplying the first equation by the number a, then subtracting it with the second equation to obtain $y=\frac{4a-8}{a^2-a-2}$. Next, PST explains that the system has no solution if *y* is not defined, and the system has many solutions if *y* is defined. PST' answer with "symbolic" MTT category 2 can be seen in Figure 7.



Data reduction and translation:
Multiply the first equation by a and we get the system $\begin{cases} ax-a(1-a)y=4a \\ ax+2y=8 \end{cases}$. Subtracting the first equation from the second equation 2 gives $-a(1-a)y-2y=4a-8$. This gives $(a^2-a-2)y=4a-8$ or $y=\frac{4a-8}{a^2-a-2}$. The system has no solution if y is undefined i.e. if $a^2-a-2=0$. So if $a=-1$ and $a=2$ the system has no solution. Furthermore, the system has many solutions if y is defined i.e. if $a^2-a-2 \neq 0$. Take a other than -1 and 2, for example $a=3$.

Figure 7. PST answer in the "symbolic" MTT Category 2

The third is the "embodied-symbolic" MTT. According to Table 2, there is only one category for this MTT. PST in this category makes a shift from embodied to symbolic MTT. As the name implies, they begin with something physical, such as the graphic representation of the equations in the

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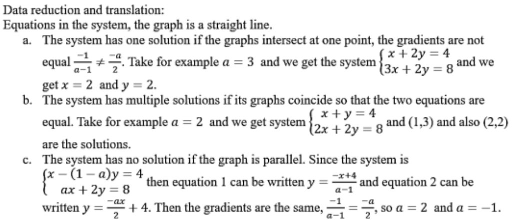
system, and then shift to symbols, operations, and properties, such as expressing the two equations in "y" as a function of "x" followed by determining the value of "a". PST answer in the "embodied-symbolic" MTT can be seen in Figure 8.

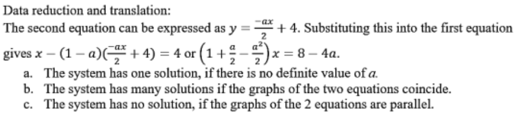


Data reduction and translation:
Equations in the system, the graph is a straight line.

a. The system has one solution if the graphs intersect at one point, the gradients are not equal $\frac{-1}{a-1} \neq \frac{-a}{2}$. Take for example $a = 3$ and we get the system $\begin{cases} x + 2y = 4 \\ 3x + 2y = 8 \end{cases}$ and we get $x = 2$ and $y = 2$.
b. The system has multiple solutions if its graphs coincide so that the two equations are equal. Take for example $a = 2$ and we get system $\begin{cases} x + y = 4 \\ 2x + 2y = 8 \end{cases}$ and (1,3) and also (2,2) are the solutions.
c. The system has no solution if the graph is parallel. Since the system is $\begin{cases} x - (1-a)y = 4 \\ ax + 2y = 8 \end{cases}$ then equation 1 can be written $y = \frac{-x+4}{a-1}$ and equation 2 can be written $y = \frac{-ax}{2} + 4$. Then the gradients are the same, $\frac{-1}{a-1} = \frac{-a}{2}$, so $a = 2$ and $a = -1$.

Figure 8. PST Answer in the "embodied-symbolic" MTT

The next is the "symbolic-embodied" MTT. According to Table 2, there is only one category for this MTT. PST in this category makes a shift from symbolic to embodied MTT. As the name implies, they begin with a focus on symbols, operations, and their properties, then shift to the physical form of the equations. PST expresses "y" as a function of "x" in the second equation, and then they shift the physical form of the equation, such as stating the relative position of the two graphs of the equation in the system. PST's answer in the "symbolic-embodied" MTT can be seen in Figure 9.

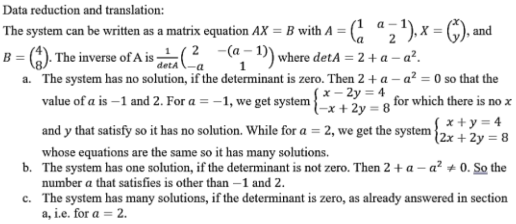


Data reduction and translation:
The second equation can be expressed as $y = \frac{-ax}{2} + 4$. Substituting this into the first equation gives $x - (1-a)(\frac{-ax}{2} + 4) = 4$ or $\left(1 + \frac{a}{2} - \frac{a^2}{2}\right)x = 8 - 4a$.

a. The system has one solution, if there is no definite value of a.
b. The system has many solutions if the graphs of the two equations coincide.
c. The system has no solution, if the graphs of the 2 equations are parallel.

Figure 9. PST answer in the "symbolic-embodied" MTT

After the "symbolic-embodied" MTT is the "formal" MTT. According to Table 2, there is only one category for this MTT. PST answers in this category are focused on the concepts needed to solve problems, such as the concept of matrix and its determinant. PST begins with writing the system of equations in the form of a matrix equation $AX = B$, and then calculating the determinant of matrix A and determining its inverse. PST's answer in the "formal" MTT can be seen in Figure 10.

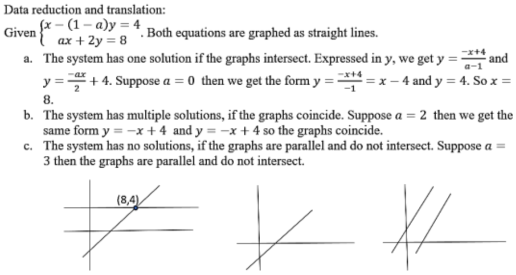


Data reduction and translation:
The system can be written as a matrix equation $AX = B$ with $A = \begin{pmatrix} 1 & a-1 \\ a & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, and $B = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$. The inverse of A is $\frac{1}{detA}\begin{pmatrix} 2 & -(a-1) \\ -a & 1 \end{pmatrix}$ where $detA = 2 + a - a^2$.

a. The system has no solution, if the determinant is zero. Then $2 + a - a^2 = 0$ so that the value of a is -1 and 2. For $a = -1$, we get system $\begin{cases} x - 2y = 4 \\ -x + 2y = 8 \end{cases}$ for which there is no x and y that satisfy so it has no solution. While for $a = 2$, we get the system $\begin{cases} x + y = 4 \\ 2x + 2y = 8 \end{cases}$ whose equations are the same so it has many solutions.
b. The system has one solution, if the determinant is not zero. Then $2 + a - a^2 \neq 0$. So the number a that satisfies is other than -1 and 2.
c. The system has many solutions, if the determinant is zero, as already answered in section a, i.e. for $a = 2$.

Figure 10. PST Answer in the "formal" MTT

The sixth is the "embodied-symbolic-embodied" MTT. According to Table 2, there is only one category for this MTT. PST in this category switches from embodied to symbolic before going back to embodied MTT. PST in this category begins with the physical form of the equations in the system (stating the graph of the equation and the relative position of the graph of two equations), shifts to symbols, operations, and their properties (calculating and manipulating symbols), and then return to the physical form (drawing graphs). PST answer in the "embodied-symbolic-embodied" MTT can be seen in Figure 11.



Data reduction and translation:
Given $\begin{cases} x - (1-a)y = 4 \\ ax + 2y = 8 \end{cases}$. Both equations are graphed as straight lines.

a. The system has one solution if the graphs intersect. Expressed in y, we get $y = \frac{-x+4}{a-1}$ and $y = \frac{-ax}{2} + 4$. Suppose $a = 0$ then we get the form $y = \frac{-x+4}{-1} = x - 4$ and $y = 4$. So $x = 8$.
b. The system has multiple solutions, if the graphs coincide. Suppose $a = 2$ then we get the same form $y = -x + 4$ and $y = -x + 4$ so the graphs coincide.
c. The system has no solutions, if the graphs are parallel and do not intersect. Suppose $a = 3$ then the graphs are parallel and do not intersect.

Figure 11. PST Answer in the "embodied-symbolic-embodied" MTT

The last MTT is the "symbolic-embodied-symbolic" MTT. According to Table 2, there is only one category for this MTT. PST in this category switches from symbolic to embodied before going back to symbolic MTT. PST in this category begins with symbols,

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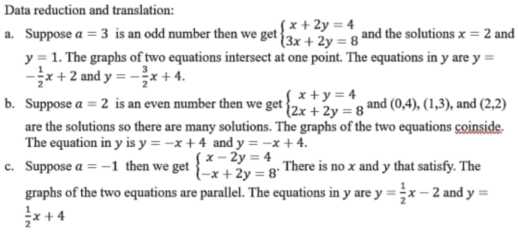
operations, and their properties (choosing the number *a* first), shifts to physical form (stating the relative position of the two-equation graphs), and then returns to symbols, operations, and their properties (expressing both equations in terms of "*y*" as a function of "*x*"). PST's answer in the "symbolic-embodied-symbolic" MTT can be seen in Figure 12.



Data reduction and translation:

a. Suppose $a = 3$ is an odd number then we get $\begin{cases} x + 2y = 4 \\ 3x + 2y = 8 \end{cases}$ and the solutions $x = 2$ and $y = 1$. The graphs of two equations intersect at one point. The equations in y are $y = -\frac{1}{2}x + 2$ and $y = -\frac{3}{2}x + 4$.

b. Suppose $a = 2$ is an even number then we get $\begin{cases} x + y = 4 \\ 2x + 2y = 8 \end{cases}$ and (0,4), (1,3), and (2,2) are the solutions so there are many solutions. The graphs of the two equations coinside. The equation in y is $y = -x + 4$ and $y = -x + 4$.

c. Suppose $a = -1$ then we get $\begin{cases} x - 2y = 4 \\ -x + 2y = 8 \end{cases}$. There is no x and y that satisfy. The graphs of the two equations are parallel. The equations in y are $y = \frac{1}{2}x - 2$ and $y = \frac{1}{2}x + 4$

Figure 12. PST Answer in the "symbolic-embodied-symbolic" MTT

Seven classifications of PST's MTT in solving a system of linear equations in two-variable problems were discovered based on the research data presentation and analysis: (1) embodied, (2) symbolic, (3) embodied-symbolic, (4) symbolic-embodied, (5) formal, (6) embodied-symbolic-embodied, and (7) symbolic-embodied-symbolic. It supports the claim of (Hailikari, 2009; Thurn et al., 2022) that different students have different prior knowledge. Further, it is also consistent with (Hailikari, 2009; Thurn et al., 2022; Witherby et al., 2023) that the diversity of prior knowledge influences the quality of student learning outcomes. The quantity of correct and wrong answers that PST had been provided when solving problems on research instruments is indicative of this. Of the 33 answers, there were only two correct answers that came from one MTT, the "formal" one, and the other six MTTs produced 31 wrong answers.

For PST with "embodied" MTT, their prior knowledge is dominated by mathematical ideas that are physical in nature, such as stating the graphic form of the equation and explaining the relative position of the two graphs. The criteria for the "embodied" world in the TWM are met by this result (Chin et al., 2022; Tall, 2008, 2020b). PSTs with this MTT are unable to connect the physical form of the equations in the system to their symbols, operations, and properties. In addition, PST struggles to make the connection between the physical form of the equations and the concepts (Stewart et al., 2019), such as the matrix and its determinants, which are necessary to solve the system of linear equations. Although the TWM are closely related theoretically (Tall, 2019, 2020b), research indicates that PSTs with "embodied" MTT are unable to relate to the "symbolic" and "formal" worlds.

For PST with "symbolic" MTT, their prior knowledge is dominated by mathematical ideas, which are procedures for obtaining solutions, such as elimination and substitution, symbol manipulation, and choosing a particular number to be substituted into the system. The criteria for the "symbolic" world in the TWM are met by this result (Chin et al., 2022; Tall, 2008, 2020b). The procedures carried out are limited to memorization (rules without reasons) (Skemp, 1976). However, when the aforementioned procedures cannot be carried out, PST makes errors and misconceptions (Hailikari, 2009; Wu, 2017). Students are unable to connect with the concepts needed (Stewart et al., 2019) to solve systems of linear equations. Students with "symbolic" LBM cannot relate to the "formal" world. Their prior knowledge hinders them from completing tasks (Södervik et al., 2022).

PST with "formal" MTT already have concepts that are well-connected in

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their prior knowledge, which they can draw upon to complete assignments (Sibgatullin et al., 2022). The essential concepts, such as the use of matrix and its determinant, are already well understood by the PST. The results of research data analysis also confirm that in PST with "formal" MTT, the TWM consisting of embodied, symbolic, and formal are well integrated (Tall, 2008, 2020a).

The other MTTs discovered in this research are mixed MTTs, i.e., "embodied-symbolic", "symbolic-embodied", "embodied-symbolic-embodied", and "symbolic-embodied-symbolic". There was a shift in the MTT of the PST from embodied to symbolic, or vice versa, but not to formal. For PST with these MTTs, their prior knowledge is dominated by mathematical ideas of a physical nature and also procedures for obtaining solutions. However, PSTs are unable to connect with the necessary concepts, such as matrices and their determinants or other necessary concepts. Even when procedures are implemented to find solutions, the procedures are carried out without understanding (Skemp, 1976) resulting in errors and misconceptions (Hailikari, 2009; Wu, 2017).

These PTTs with mixed MTTs, along with "embodied" and "symbolic", need to get past errors and misconceptions (Hailikari, 2009; Wu, 2017) through meaningful learning (Kärki et al., 2018; Mayer, 2002) that strengthen the connection between their prior knowledge and what they need to solve problems (Wettergren, 2022), facilitate to understand how to use mathematical symbols (Kieran, 2022; Levin & Walkoe, 2022), facilitate and encourage to shift from contextual mathematics to structural mathematics (Sibgatullin et al., 2022). From PSTs point of view, they should develop algebraic thinking skills (Sibgatullin et al., 2022; Wettergren, 2022) through change and develop their cognitive structures (Aarto-Pesonen & Piirainen, 2020; Fazio et al., 2016), use their prior knowledge in various new situations (Ahmed et al., 2019), create new meaning by integrating their prior knowledge (Kärki et al., 2018; Mayer, 2002) and be independent to achieve maximum results (Vargas-Hernández & Vargas-González, 2022).

CONCLUSIONS AND SUGGESTIONS

This research yielded seven classifications of PST MTT in solving a system of linear equations in two variables: (1) embodied, (2) symbolic, (3) embodied-symbolic, (4) symbolic-embodied, (5) formal, (6) embodied-symbolic-embodied, and (7) symbolic-embodied-symbolic. Table 2 provides descriptions of the seven MTTs mentioned above.

PST with "embodied" and "symbolic" MTT, or a combination of both, have prior knowledge of a system of linear equations, which is not sufficient to complete the assignment. The implemented strategy to find solutions is still unconnected to the required concepts. To get around this, learning should be designed so that (1) PST can strengthen the connection between their prior knowledge and what they need to solve problems involving different system of linear equations, (2) can facilitate PST to understand how to use mathematical symbols, and (3) can facilitate and encourage PST to shift from contextual mathematics to structural mathematics. With this kind of learning, PST will be able to get past errors and misconceptions when finishing mathematical assignments,

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particularly system of linear equations in two variables.

Additionally, learning must be designed so that PST can (1) encourage cognitive change and development in their cognitive structures, (2) use their prior knowledge in various new situations, (3) create new meaning by integrating their prior knowledge; and (4) encourage PST independence to achieve maximum results. In short, the aforementioned abilities will be achieved if learning is designed in the form of meaningful learning and learning that facilitates PST to be able to develop algebraic thinking skills.

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